INTEGER DESIGN OF SOLUTIONS OF ONE OF THE DIOPHANTINE EQUATIONS $\boldsymbol{\alpha}(\mathbf{X^4+Y^4})(\mathbf{P^2+Q^2+R^2+S^2}) = (\mathbf{T^2+U^2})(\mathbf{C^2-D^2})(\mathbf{Z^2-W^2})\mathbf{P}^{\boldsymbol{\beta}}$ WITH X<Y<W<Z and P is ODD, $\boldsymbol{\alpha>0,\beta>0}$

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Abstract:

In this paper focused to study to find integer design of solutions Diophantine Equation $\boldsymbol{\alpha}(X^4+Y^4)(p^2+q^2+r^2+s^2)=(t^2+u^2)(C^2-D^2)(Z^2-W^2)P^{\beta}$ with $\alpha>0,\beta>0$ with Mathematical induction method for $\beta=1,2,3,4,\dots$ *and so on*. Having integer design of solutions for $\beta>2$ is

x $=k^n, y=k^{n+1}, z=k^{n+3}, w=k^{n+2}, p=k^n$, $C=\left(\left(\frac{1+k^4}{2}\right)^2+1\right)n$,

$D=\left(\left(\frac{1+k^4}{2}\right)^2-1\right)n, \alpha=(1+k^4)(k^6-k^4)k^{(\beta-2)n}n^2, q=p+1\ ,\ r=\frac{p^2-1}{2}$,

$s=\left(\frac{p+1}{2}\right)^2-1\ ,\ t=\frac{p^2+1}{2},\ u=\left(\frac{p+1}{2}\right)^2+1$ .whenever $1+k^4$ is even, p is odd

and

x $=k^n, y=k^{n+1}, z=k^{n+3}, w=k^{n+2}, p=k^n, C=\left(\frac{(1+k^4)^2+1}{2})\right)n, D=\left(\frac{(1+k^4)^2-1}{2}\right)n,\ \alpha=(1+k^4)(k^6-k^4)k^{(\beta-2)n}n^2, q=p+1\ ,\ r=\frac{p^2-1}{2}, s=\left(\frac{p+1}{2}\right)^2-1\ ,\ t=\frac{p^2+1}{2}$,

$u=\left(\frac{p+1}{2}\right)^2+1$. if $1+k^4$ and p is odd. Here k must be positive integer.

I. Introduction:

Diophantine equations, which are polynomial equations restricted to integer solutions, occupy a central place in algebraic number theory. Within this broad framework, exponential and higher-degree forms—including quintic Diophantine equations—extend their significance beyond pure mathematics into applied scientific domains.

II. Results & Discussions:

Proportion 1: A Study on Diophantine Equation

$\boldsymbol{\alpha}(X^4 + Y^4)(p^2 + q^2 + r^2 + s^2) = (t^2 + u^2)(C^2 - D^2)(Z^2 - W^2)P.$

Clearly, we know that from Reference [1], $p^2 + q^2 + r^2 + s^2 = t^2 + u^2$ having different sets of integer solutions is illustrated below:

if p is odd then $q = p + 1$, $r = \frac{p^2-1}{2}$, $s = \left(\frac{p+1}{2}\right)^2 - 1$, $t = \frac{p^2+1}{2}$, $u = \left(\frac{p+1}{2}\right)^2 + 1$.

It follows that we go to prove that find integer design of solutions of

$$\boldsymbol{\alpha}(X^4 + Y^4) = (C^2 - D^2)(Z^2 - W^2)P$$

Let x = $k^n, y = k^{n+1}, z = k^{n+3}, w = k^{n+2}, p = k^{2n}, q = p + 1$, $r = \frac{p^2-1}{2}$, $s = \left(\frac{p+1}{2}\right)^2 - 1$, $t = \frac{p^2+1}{2}$, $u = \left(\frac{p+1}{2}\right)^2 + 1$.

Consider $\alpha(X^4 + Y^4) = \alpha k^{4n}(1 + k^4)$

Again consider $(Z^2 - W^2)P = k^{4n}(k^6 - k^4)$.

It follows that $\alpha(X^4 + Y^4) = (C^2 - D^2)(Z^2 - W^2)P$ implies that

$\alpha k^{4n}(1 + k^4) = (C^2 - D^2)k^{4n}(k^6 - k^4)$ implies $\alpha(1 + k^4) = (C^2 - D^2)(k^6 - k^4)$. Solve for α, whenever $(1 + k^4, D, C)$ becomes a Pythagorean Triplet.

From the References, we know that

if r is an even number, then (r, $\left(\frac{r}{2}\right)^2 - 1$, $\left(\frac{r}{2}\right)^2 + 1$) is a Pythagorean triplet.

if r is an odd number, then (r, $\frac{r^2-1}{2}$, $\frac{r^2+1}{2}$) is a Pythagorean triplet.

It implies that $(1 + k^4, D, C)$ becomes a Pythagorean Triplet depends on $1 + k^4$ is it odd or even.

Case 1: If $1 + k^4$ is even, then ($1 + k^4$, $\left(\frac{1+k^4}{2}\right)^2 - 1$, $\left(\frac{1+k^4}{2}\right)^2 + 1$) is a Pythagorean triplet. It follows that

$\alpha(1 + k^4) = (C^2 - D^2)(k^6 - k^4)$ and solve for α, whenever $(1 + k^4, D, C)$ becomes a Pythagorean Triplet with $C = \left(\left(\frac{1+k^4}{2}\right)^2 + 1\right)n$, $D = \left(\left(\frac{1+k^4}{2}\right)^2 - 1\right)n$

$C^2 - D^2 = (1 + k^4)^2 n^2$ and hence $\alpha = (1 + k^4)(k^6 - k^4)n^2$.

Hence, **we obtain $\boldsymbol{\alpha}(X^4 + Y^4)(p^2 + q^2 + r^2 + s^2) = (t^2 + u^2)(C^2 - D^2)(Z^2 - W^2)P$**

having integer design of solution is x = $\boldsymbol{k^n, y = k^{n+1}, z = k^{n+3}, w = k^{n+2}, p = k^{2n}}$,

$\boldsymbol{C = \left(\left(\frac{1+k^4}{2}\right)^2 + 1\right)n,\ D = \left(\left(\frac{1+k^4}{2}\right)^2 - 1\right)n, \alpha = (1+k^4)(k^6-k^4)n^2}, q = \mathbf{p+1}$ **,**

$\mathbf{r = \frac{p^2-1}{2}}$**, s=** $\mathbf{\left(\frac{p+1}{2}\right)^2 - 1}$ **, t =** $\mathbf{\frac{p^2+1}{2}}$**,** $\mathbf{u = \left(\frac{p+1}{2}\right)^2 + 1}$ **when ever p is odd.**

Case 2: If $1+k^4$ is odd, then ($1+k^4, \frac{(1+k^4)^2-1}{2}, \frac{(1+k^4)^2+1}{2}$) is a Pythagorean triplet. It follows that

$\alpha(1+k^4) = (C^2-D^2)(k^6-k^4)$ and solve for α, whenever $(1+k^4, D, C)$ becomes a Pythagorean Triplet with $C = \left(\frac{(1+k^4)^2+1}{2})\right)n$, $D = \left(\frac{(1+k^4)^2-1}{2}\right)n$. Hence

$C^2 - D^2 = (1+k^4)^2 n^2$ and hence $\alpha = (1+k^4)(k^6-k^4)n^2$.

Hence**, we obtain** $\boldsymbol{\alpha(X^4+Y^4) = (C^2-D^2)(Z^2-W^2)P}$**.**

Having an integer design of solution is x = $\boldsymbol{k^n, y = k^{n+1}, z = k^{n+3}, w = k^{n+2}, p = k^{2n}}$**,**

$\boldsymbol{C = \left(\frac{(1+k^4)^2+1}{2})\right)n, D = \left(\frac{(1+k^4)^2-1}{2}\right)n,\ \alpha = (1+k^4)(k^6-k^4)}$**.**

E.g. 1: Suppose k = 2 then $1+k^4$ = 17, is odd; Having an integer design of solution is

x = $2^n, y = 2^{n+1}, z = 2^{n+3}, w = 2^{n+2}, p = 2^{2n}$,

$C = \left(\frac{(1+2^4)^2+1}{2})\right)n = 145n, D = \left(\frac{(1+2^4)^2-1}{2}\right) = 144nn$

$C^2 - D^2 = (1+2^4)^2 n^2,\ \alpha = (1+2^4)(2^6-2^4)n^2$.

Suppose n = 1; then x = 2, $y = 4, z = 16, w = 8, p = 4$,

$C = \left(\frac{(1+2^4)^2+1}{2})\right) = 145, D = \left(\frac{(1+2^4)^2-1}{2}\right) = 144$

$C^2 - D^2 = (1+2^4)^2 = 289,\ \alpha = (1+2^4)(2^6-2^4) = 816$.

Consider **LHS=** $\boldsymbol{\alpha(X^4+Y^4)}$ **=816**$\mathbf{(2^4+4^4) = 221952}$**.**

RHS $= \boldsymbol{(C^2-D^2)(Z^2-W^2)P = \mathbf{289 * (16^2-8^2) * 4 = 221952}}$**.**

E.g. 2: Suppose k = 3 then $1+k^4$ = 82, is even; Having an integer design of solution is

x = $3^n, y = 3^{n+1}, z = 3^{n+3}, w = 3^{n+2}, p = 3^{2n}$,

$C = \left(\left(\frac{1+3^4}{2}\right)^2 + 1\right)n = 1681n,\ D = \left(\left(\frac{1+3^4}{2}\right)^2 - 1\right)n = 1600n$

$C^2 - D^2 = (1+3^4)^2 n^2 = 6724n,\ \alpha = (1+3^4)(3^6-3^4)n^2 = 53136n^2$.

Suppose n = 1; then x = 3, $y = 9, z = 81, w = 27, p = 9, C^2 - D^2 = 6724,\ \alpha = 53136$

Consider **LHS=** $\boldsymbol{\alpha(X^4+Y^4)}$ **=53136**$\mathbf{(3^4+9^4) = 352929312}$**.**

RHS $= \boldsymbol{(C^2-D^2)(Z^2-W^2)P = \mathbf{6724 * (81^2-27^2) * 9 = 352929312}}$**.**

Proportion 2: A Study on Diophantine Equation

$$\boldsymbol{\alpha}(X^4+Y^4)(p^2+q^2+r^2+s^2)=(t^2+u^2)(C^2-D^2)(Z^2-W^2)P^2$$

Clearly, we know that from Reference [1], $p^2+q^2+r^2+s^2=t^2+u^2$ having different sets of integer solutions is illustrated below:

if p is odd then $q=p+1$, $r=\frac{p^2-1}{2}$, $s=\left(\frac{p+1}{2}\right)^2-1$, $t=\frac{p^2+1}{2}$, $u=\left(\frac{p+1}{2}\right)^2+1$.

Now we can find integer design of solutions of $\boldsymbol{\alpha}(X^4+Y^4)=(C^2-D^2)(Z^2-W^2)P^2$

Let x = $k^n, y=k^{n+1}, z=k^{n+3}, w=k^{n+2}, p=k^n$

Consider $\alpha(X^4+Y^4)=\alpha k^{4n}(1+k^4)$

Again consider $(Z^2-W^2)\boldsymbol{P^2}=k^{4n}(k^6-k^4)$.

It follows that $\alpha(X^4+Y^4)=(C^2-D^2)(Z^2-W^2)\boldsymbol{P^2}$ implies that

$\alpha k^{4n}(1+k^4)=(C^2-D^2)k^{4n}(k^6-k^4)$ implies $\alpha(1+k^4)=(C^2-D^2)(k^6-k^4)$. Solve for α, whenever $(1+k^4, D, C)$ becomes a Pythagorean Triplet.

From the References, we know that

if r is an even number, then (r, $\left(\frac{r}{2}\right)^2-1$, $\left(\frac{r}{2}\right)^2+1$) is a Pythagorean triplet.

if r is an odd number, then (r, $\frac{r^2-1}{2}$, $\frac{r^2+1}{2}$) is a Pythagorean triplet.

It implies that $(1+k^4, D, C)$ becomes a Pythagorean Triplet depends on $1+k^4$ is it odd or even.

Case 1: If $1+k^4$ is even, then ($1+k^4$, $\left(\frac{1+k^4}{2}\right)^2-1$, $\left(\frac{1+k^4}{2}\right)^2+1$) is a Pythagorean triplet. It follows that

$\alpha(1+k^4)=(C^2-D^2)(k^6-k^4)$ and solve for α, whenever $(1+k^4, D, C)$ becomes a Pythagorean Triplet with $C=\left(\left(\frac{1+k^4}{2}\right)^2+1\right)n$, $D=\left(\left(\frac{1+k^4}{2}\right)^2-1\right)n$

$C^2-D^2=(1+k^4)^2n^2$ and hence $\alpha=(1+k^4)(k^6-k^4)n^2$.

Hence, **we obtain** $\boldsymbol{\alpha}(X^4+Y^4)(p^2+q^2+r^2+s^2)=(t^2+u^2)(C^2-D^2)(Z^2-W^2)P^2$

having integer design of solution is x = $\boldsymbol{k^n, y=k^{n+1}, z=k^{n+3}, w=k^{n+2}, p=k^n}$, $\boldsymbol{C=\left(\left(\frac{1+k^4}{2}\right)^2+1\right)n}$, $\boldsymbol{D=\left(\left(\frac{1+k^4}{2}\right)^2-1\right)n, \alpha=(1+k^4)(k^6-k^4)n^2}$.

Case 2: If $1+k^4$ is odd, then ($1+k^4$, $\frac{(1+k^4)^2-1}{2}$, $\frac{(1+k^4)^2+1}{2}$) is a Pythagorean triplet. It follows that $\alpha(1+k^4)=(C^2-D^2)(k^6-k^4)$ and solve for α, whenever $(1+k^4, D, C)$ becomes a Pythagorean Triplet with $C=\left(\frac{(1+k^4)^2+1}{2})\right)n$, $D=\left(\frac{(1+k^4)^2-1}{2}\right)n$.

Hence $C^2 - D^2 = (1+k^4)^2 n^2$ and hence $\alpha = (1+k^4)(k^6 - k^4)n^2$.

Hence, **we obtain** $\boldsymbol{\alpha}(X^4 + Y^4)(p^2 + q^2 + r^2 + s^2) = (t^2 + u^2)(C^2 - D^2)(Z^2 - W^2)P^2$, **having an integer design of solution is x =** $\boldsymbol{k^n, y = k^{n+1}, z = k^{n+3}, w = k^{n+2}, p = k^n}$,

$\boldsymbol{C = \left(\frac{(1+k^4)^2+1}{2})\right)n, D = \left(\frac{(1+k^4)^2-1}{2}\right)n,\ \alpha = (1+k^4)(k^6-k^4)n^2.}$

E.g. 1: Suppose k = 2 then $1 + k^4$ = 17, is odd; Having an integer design of solution is

x = $2^n, y = 2^{n+1}, z = 2^{n+3}, w = 2^{n+2}, p = 2^n$,

$C = \left(\frac{(1+2^4)^2+1}{2})\right)n = 145n, D = \left(\frac{(1+2^4)^2-1}{2}\right) = 144nn$

$C^2 - D^2 = (1+2^4)^2 n^2,\ \alpha = (1+2^4)(2^6 - 2^4)n^2$.

Suppose n = 1;

then x = $2, y = 4, z = 16, w = 8, p = 2, C = \left(\frac{(1+2^4)^2+1}{2})\right) = 145, D = \left(\frac{(1+2^4)^2-1}{2}\right) = 144$

$C^2 - D^2 = (1+2^4)^2 = 289,\ \alpha = (1+2^4)(2^6 - 2^4) = 816.$

Consider **LHS=** $\boldsymbol{\alpha(X^4 + Y^4)}$ **=816**$\boldsymbol{(2^4 + 4^4) = 221952.}$

RHS $= \boldsymbol{(C^2 - D^2)(Z^2 - W^2)P^2 = 289 * (16^2 - 8^2) * 4 = 221952.}$

E.g. 2: Suppose k = 3 then $1 + k^4$ = 82, is even; Having an integer design of solution is

x = $3^n, y = 3^{n+1}, z = 3^{n+3}, w = 3^{n+2}, p = 3^{2n}, C = \left(\left(\frac{1+3^4}{2}\right)^2 + 1\right)n = 1681n$,

$D = \left(\left(\frac{1+3^4}{2}\right)^2 - 1\right)n = 1600n,\ C^2 - D^2 = (1+3^4)^2 n^2 = 6724n,$

$\alpha = (1+3^4)(3^6 - 3^4)n^2 = 53136n^2.$

Suppose n = 1;

then x = $3, y = 9, z = 81, w = 27, p = 3, C = \left(\left(\frac{1+3^4}{2}\right)^2 + 1\right) = 1681,$

$D = \left(\left(\frac{1+3^4}{2}\right)^2 - 1\right) = 1600,\ C^2 - D^2 = 6724,\ \alpha = 53136$

Consider **LHS=** $\boldsymbol{\alpha(X^4 + Y^4)}$ **=53136**$\boldsymbol{(3^4 + 9^4) = 352929312.}$

RHS $= \boldsymbol{(C^2 - D^2)(Z^2 - W^2)P^2 = 6724 * (81^2 - 27^2) * 9 = 352929312.}$

Proportion 3: A Study on Diophantine Equation

$$\boldsymbol{\alpha}(X^4 + Y^4)(p^2 + q^2 + r^2 + s^2) = (t^2 + u^2)(C^2 - D^2)(Z^2 - W^2)P^3$$

Clearly, we know that from Reference [1], $p^2 + q^2 + r^2 + s^2 = t^2 + u^2$ having different sets of integer solutions is illustrated below:

if p is odd then $q = p + 1$, $r = \frac{p^2-1}{2}$, $s = \left(\frac{p+1}{2}\right)^2 - 1$, $t = \frac{p^2+1}{2}$, $u = \left(\frac{p+1}{2}\right)^2 + 1$.

Now we can find integer design of solutions of $\boldsymbol{\alpha}(X^4 + Y^4) = (C^2 - D^2)(Z^2 - W^2)P^3$

Let x = $k^n, y = k^{n+1}, z = k^{n+3}, w = k^{n+2}, p = k^n$

Consider $\alpha(X^4 + Y^4) = \alpha k^{4n}(1 + k^4)$

Again consider $(Z^2 - W^2)\boldsymbol{P^3} = k^{5n}(k^6 - k^4)$.

It follows that $\alpha(X^4 + Y^4) = (C^2 - D^2)(Z^2 - W^2)\boldsymbol{P^2}$ implies that

$\alpha k^{4n}(1 + k^4) = (C^2 - D^2)k^{5n}(k^6 - k^4)$ implies $\alpha(1 + k^4) = (C^2 - D^2)(k^6 - k^4)k^n$. Solve for α, whenever $(1 + k^4, D, C)$ becomes a Pythagorean Triplet.

From the References, we know that

if r is an even number, then (r, $\left(\frac{r}{2}\right)^2 - 1, \left(\frac{r}{2}\right)^2 + 1$) is a Pythagorean triplet.

if r is an odd number, then (r, $\frac{r^2-1}{2}, \frac{r^2+1}{2}$) is a Pythagorean triplet.

It implies that $(1 + k^4, D, C)$ becomes a Pythagorean Triplet depends on $1 + k^4$ is it odd or even.

Case 1: If $1 + k^4$ is even, then ($1 + k^4, \left(\frac{1+k^4}{2}\right)^2 - 1, \left(\frac{1+k^4}{2}\right)^2 + 1$) is a Pythagorean triplet. It follows that

$\alpha(1 + k^4) = (C^2 - D^2)(k^6 - k^4)k^n$ and solve for α, whenever $(1 + k^4, D, C)$ becomes a Pythagorean Triplet with $C = \left(\left(\frac{1+k^4}{2}\right)^2 + 1\right)n$, $D = \left(\left(\frac{1+k^4}{2}\right)^2 - 1\right)n$

$C^2 - D^2 = (1 + k^4)^2 n^2$ and hence $\alpha = (1 + k^4)(k^6 - k^4)k^n n^2$.

Hence, **we obtain** $\boldsymbol{\alpha}(X^4 + Y^4)(p^2 + q^2 + r^2 + s^2) = (t^2 + u^2)(C^2 - D^2)(Z^2 - W^2)P^3$

having integer design of solution is x = $\boldsymbol{k^n, y = k^{n+1}, z = k^{n+3}, w = k^{n+2}, p = k^n, C = \left(\left(\frac{1+k^4}{2}\right)^2 + 1\right)n, D = \left(\left(\frac{1+k^4}{2}\right)^2 - 1\right)n, \alpha = (1 + k^4)(k^6 - k^4)}k^n\boldsymbol{n^2}$.

Case 2: If $1 + k^4$ is odd, then ($1 + k^4, \frac{(1+k^4)^2-1}{2}, \frac{(1+k^4)^2+1}{2}$) is a Pythagorean triplet. It follows that $\alpha(1 + k^4) = (C^2 - D^2)(k^6 - k^4)k^n$ and solve for α, whenever $(1 + k^4, D, C)$ becomes a Pythagorean Triplet with $C = \left(\frac{(1+k^4)^2+1}{2}\right)n$, $D = \left(\frac{(1+k^4)^2-1}{2}\right)n$. Hence

$C^2 - D^2 = (1 + k^4)^2 n^2$ and hence $\alpha = (1 + k^4)(k^6 - k^4)k^n n^2$.

Hence**, we obtain** $\boldsymbol{\alpha}(X^4 + Y^4)(p^2 + q^2 + r^2 + s^2) = (t^2 + u^2)(C^2 - D^2)(Z^2 - W^2)P^3$ **having an integer design of solution is x =** $\boldsymbol{k^n, y = k^{n+1}, z = k^{n+3}, w = k^{n+2}, p = k^n}$**,**

$\boldsymbol{C = \left(\frac{(1+k^4)^2+1}{2}\right)n, D = \left(\frac{(1+k^4)^2-1}{2}\right)n, \alpha = (1 + k^4)(k^6 - k^4)}k^n\boldsymbol{n^2}$**.**

E.g. 1: Suppose k = 2 then $1 + k^4$ = 17, is odd; Having an integer design of solution is

x = $2^n, y = 2^{n+1}, z = 2^{n+3}, w = 2^{n+2}, p = 2^n$,

$C = \left(\frac{(1+2^4)^2+1}{2})\right)n = 145n, D = \left(\frac{(1+2^4)^2-1}{2}\right) = 144nn$

$C^2 - D^2 = (1 + 2^4)^2 n^2,\ \alpha = (1 + 2^4)(2^6 - 2^4) * 2^n * n^2$.

Suppose n = 1;

then x = $2, y = 4, z = 16, w = 8, p = 2, C = \left(\frac{(1+2^4)^2+1}{2})\right) = 145, D = \left(\frac{(1+2^4)^2-1}{2}\right) = 144$

$C^2 - D^2 = (1 + 2^4)^2 = 289,\ \alpha = (1 + 2^4)(2^6 - 2^4) * 2 = 1632.$

Consider **LHS=** $\boldsymbol{\alpha(X^4 + Y^4)}$ =**1632***$\mathbf{(2^4 + 4^4) = 443904.}$

RHS $= \boldsymbol{(C^2 - D^2)(Z^2 - W^2)P^3 =\ 289 * (16^2 - 8^2) * 8 = 443904.}$

E.g. 2: Suppose k = 3 then $1 + k^4$ = 82, is even; Having an integer design of solution is

x = $3^n, y = 3^{n+1}, z = 3^{n+3}, w = 3^{n+2}, p = 3^n, C = \left(\left(\frac{1+3^4}{2}\right)^2 + 1\right)n = 1681n$,

$D = \left(\left(\frac{1+3^4}{2}\right)^2 - 1\right)n = 1600n,\ C^2 - D^2 = (1 + 3^4)^2 n^2 = 6724n,$

$\alpha = (1 + 3^4)(3^6 - 3^4)3^n n^2 = 53136 * 3^n * n^2.$

Suppose n = 1;

then x = $3, y = 9, z = 81, w = 27, p = 3, C = \left(\left(\frac{1+3^4}{2}\right)^2 + 1\right) = 1681,$

$D = \left(\left(\frac{1+3^4}{2}\right)^2 - 1\right) = 1600,\ C^2 - D^2 = 6724,\ \alpha = 159408$

Consider **LHS=** $\boldsymbol{\alpha(X^4 + Y^4)} = 159408\mathbf{(3^4 + 9^4) = 1058787936.}$

RHS $= \boldsymbol{(C^2 - D^2)(Z^2 - W^2)P^3 =\ 6724 * (81^2 - 27^2) * 27 = 1058787936.}$

Proportion 4: A Study on Diophantine Equation

$\boldsymbol{\alpha}(X^4 + Y^4)(p^2 + q^2 + r^2 + s^2) = (t^2 + u^2)(C^2 - D^2)(Z^2 - W^2)P^4$

Clearly, we know that from Reference [1], $p^2 + q^2 + r^2 + s^2 = t^2 + u^2$ having different sets of integer solutions is illustrated below:

if p is odd then $q = p + 1$, $r = \frac{p^2-1}{2}$, $s = \left(\frac{p+1}{2}\right)^2 - 1$, $t = \frac{p^2+1}{2}$, $u = \left(\frac{p+1}{2}\right)^2 + 1$.

Now we can find integer design of solutions of $\boldsymbol{\alpha}(X^4 + Y^4) = (C^2 - D^2)(Z^2 - W^2)P^4$

Let x = $k^n, y = k^{n+1}, z = k^{n+3}, w = k^{n+2}, p = k^n$

Consider $\alpha(X^4 + Y^4) = \alpha k^{4n}(1 + k^4)$. Again consider $(Z^2 - W^2)\boldsymbol{P^4} = k^{6n}(k^6 - k^4)$.

It follows that $\boldsymbol{\alpha}(X^4+Y^4)(p^2+q^2+r^2+s^2)=(t^2+u^2)(C^2-D^2)(Z^2-W^2)P^4$ implies that $\alpha k^{4n}(1+k^4)=(C^2-D^2)k^{6n}(k^6-k^4)$ implies

$\alpha(1+k^4)=(C^2-D^2)(k^6-k^4)k^{2n}$. Solve for α, whenever $(1+k^4,D,C)$ becomes a Pythagorean Triplet.

From the References, we know that

if r is an even number, then (r, $\left(\frac{r}{2}\right)^2-1$, $\left(\frac{r}{2}\right)^2+1$) is a Pythagorean triplet.

if r is an odd number, then (r, $\frac{r^2-1}{2}$, $\frac{r^2+1}{2}$) is a Pythagorean triplet.

It implies that $(1+k^4,D,C)$ becomes a Pythagorean Triplet depends on $1+k^4$ is it odd or even.

Case 1: If $1+k^4$ is even, then ($1+k^4$, $\left(\frac{1+k^4}{2}\right)^2-1$, $\left(\frac{1+k^4}{2}\right)^2+1$) is a Pythagorean triplet. It follows that

$\alpha(1+k^4)=(C^2-D^2)(k^6-k^4)k^n$ and solve for α, whenever $(1+k^4,D,C)$ becomes a Pythagorean Triplet with $C=\left(\left(\frac{1+k^4}{2}\right)^2+1\right)n$, $D=\left(\left(\frac{1+k^4}{2}\right)^2-1\right)n$

$C^2-D^2=(1+k^4)^2n^2$ and hence $\alpha=(1+k^4)(k^6-k^4)k^{2n}n^2$.

Hence, **we obtain** $\boldsymbol{\alpha}(X^4+Y^4)(p^2+q^2+r^2+s^2)=(t^2+u^2)(C^2-D^2)(Z^2-W^2)P^4$

having integer design of solution is x $=\boldsymbol{k^n},\boldsymbol{y}=\boldsymbol{k^{n+1}},\boldsymbol{z}=\boldsymbol{k^{n+3}},\boldsymbol{w}=\boldsymbol{k^{n+2}},\boldsymbol{p}=\boldsymbol{k^n}$, $\boldsymbol{C}=\left(\left(\frac{\mathbf{1}+\boldsymbol{k^4}}{\mathbf{2}}\right)^{\mathbf{2}}+\mathbf{1}\right)\boldsymbol{n}$, $\boldsymbol{D}=\left(\left(\frac{\mathbf{1}+\boldsymbol{k^4}}{\mathbf{2}}\right)^{\mathbf{2}}-\mathbf{1}\right)\boldsymbol{n},\boldsymbol{\alpha}=(\mathbf{1}+\boldsymbol{k^4})(\boldsymbol{k^6}-\boldsymbol{k^4})k^{2n}\boldsymbol{n^2}$.

Case 2: If $1+k^4$ is odd, then ($1+k^4$, $\frac{(1+k^4)^2-1}{2}$, $\frac{(1+k^4)^2+1}{2}$) is a Pythagorean triplet. It follows that

$\alpha(1+k^4)=(C^2-D^2)(k^6-k^4)k^n$ and solve for α, whenever $(1+k^4,D,C)$ becomes a Pythagorean Triplet with $C=\left(\frac{(1+k^4)^2+1}{2})\right)n$, $D=\left(\frac{(1+k^4)^2-1}{2}\right)n$. Hence

$C^2-D^2=(1+k^4)^2n^2$ and hence $\alpha=(1+k^4)(k^6-k^4)k^{2n}n^2$.

Hence**, we obtain** $\boldsymbol{\alpha}(X^4+Y^4)(p^2+q^2+r^2+s^2)=(t^2+u^2)(C^2-D^2)(Z^2-W^2)P^4$,

having an integer design of solution is x $=\boldsymbol{k^n},\boldsymbol{y}=\boldsymbol{k^{n+1}},\boldsymbol{z}=\boldsymbol{k^{n+3}},\boldsymbol{w}=\boldsymbol{k^{n+2}},\boldsymbol{p}=\boldsymbol{k^n}$,

$\boldsymbol{C}=\left(\frac{(\mathbf{1}+\boldsymbol{k^4})^{\mathbf{2}}+\mathbf{1}}{\mathbf{2}})\right)\boldsymbol{n},\boldsymbol{D}=\left(\frac{(\mathbf{1}+\boldsymbol{k^4})^{\mathbf{2}}-\mathbf{1}}{\mathbf{2}}\right)\boldsymbol{n},\ \boldsymbol{\alpha}=(\mathbf{1}+\boldsymbol{k^4})(\boldsymbol{k^6}-\boldsymbol{k^4})k^{2n}\boldsymbol{n^2}$**.**

E.g. 1: Suppose k = 2 then $1+k^4$ = 17, is odd; Having an integer design of solution is

x = $2^n,y=2^{n+1},z=2^{n+3},w=2^{n+2},p=2^n$,

$C = \left(\frac{(1+2^4)^2+1}{2})\right)n = 145n$, $D = \left(\frac{(1+2^4)^2-1}{2}\right) = 144n$:$C^2 - D^2 = (1+2^4)^2n^2$,

$\alpha = (1+2^4)(2^6 - 2^4) * 2^{2n} * n^2$.

Suppose n = 1;

then x = $2, y = 4, z = 16, w = 8, p = 2, C = \left(\frac{(1+2^4)^2+1}{2})\right) = 145, D = \left(\frac{(1+2^4)^2-1}{2}\right) = 144$

$C^2 - D^2 = (1+2^4)^2 = 289,\ \alpha = (1+2^4)(2^6 - 2^4) * 4 = 3264$.

Consider **LHS=** $\boldsymbol{\alpha}(\boldsymbol{X^4} + \boldsymbol{Y^4}) = 3264*(\mathbf{2^4} + \mathbf{4^4}) = \mathbf{887808}$.

RHS $= (\boldsymbol{C^2} - \boldsymbol{D^2})(\boldsymbol{Z^2} - \boldsymbol{W^2})\boldsymbol{P^4} = \mathbf{289} * (\mathbf{16^2} - \mathbf{8^2}) * \mathbf{16} = \mathbf{887808}$**.**

E.g. 2: Suppose k = 3 then $1 + k^4$ = 82, is even; Having an integer design of solution is

x = $3^n, y = 3^{n+1}, z = 3^{n+3}, w = 3^{n+2}, p = 3^n, C = \left(\left(\frac{1+3^4}{2}\right)^2 + 1\right)n = 1681n$,

$D = \left(\left(\frac{1+3^4}{2}\right)^2 - 1\right)n = 1600n,\ C^2 - D^2 = (1+3^4)^2n^2 = 6724n$,

$\alpha = (1+3^4)(3^6 - 3^4)3^{2n}n^2 = 53136 * 3^{2n} * n^2$.

Suppose n = 1; then x = $3, y = 9, z = 81, w = 27, p = 3, C = \left(\left(\frac{1+3^4}{2}\right)^2 + 1\right) = 1681$,

$D = \left(\left(\frac{1+3^4}{2}\right)^2 - 1\right) = 1600,\ C^2 - D^2 = 6724,\ \alpha = 478224$

Consider **LHS=** $\boldsymbol{\alpha}(\boldsymbol{X^4} + \boldsymbol{Y^4}) = 478224 * (\mathbf{3^4} + \mathbf{9^4}) = \mathbf{3176363808}$.

RHS $= (\boldsymbol{C^2} - \boldsymbol{D^2})(\boldsymbol{Z^2} - \boldsymbol{W^2})\boldsymbol{P^4} = \mathbf{6724} * (\mathbf{81^2} - \mathbf{27^2}) * \mathbf{81} = \mathbf{3176363808}$**.**

Main result:

A STUDY ON

$\boldsymbol{\alpha}(X^4 + Y^4)(p^2 + q^2 + r^2 + s^2) = (t^2 + u^2)(C^2 - D^2)(Z^2 - W^2)P^\beta$ with $\alpha > 0, \beta > 0$.

Clearly, we know that from Reference [1], $p^2 + q^2 + r^2 + s^2 = t^2 + u^2$ having different sets of integer solutions is illustrated below:

if p is odd then $q = p + 1$, $r = \frac{p^2-1}{2}$, $s = \left(\frac{p+1}{2}\right)^2 - 1$, $t = \frac{p^2+1}{2}$, $u = \left(\frac{p+1}{2}\right)^2 + 1$.

Now we can find integer design of solutions of $\boldsymbol{\alpha}(X^4 + Y^4) = (C^2 - D^2)(Z^2 - W^2)P^\beta$

Having integer design of solutions for $\beta > 2$ is

x = $k^n, y = k^{n+1}, z = k^{n+3}, w = k^{n+2}, p = k^n$, $C = \left(\left(\frac{1+k^4}{2}\right)^2 + 1\right)n$,

$D = \left(\left(\frac{1+k^4}{2}\right)^2 - 1\right)n, \alpha = (1+k^4)(k^6-k^4)k^{(\beta-2)n}n^2$, If $1+k^4$ is even and

x $= k^n, y = k^{n+1}, z = k^{n+3}, w = k^{n+2}, p = k^n$,

$C = \left(\frac{(1+k^4)^2+1}{2})\right)n, D = \left(\frac{(1+k^4)^2-1}{2}\right)n,\ \alpha = (1+k^4)(k^6-k^4)k^{(\beta-2)n}n^2$ if $1+k^4$ is odd.

Conclusion:

This equation generalizes classical Diophantine problems, blending sums of fourth powers with multiplicative factorizations. While challenging, targeted parametrization and modular analysis can yield solutions. Future work may classify solutions for specific α, β or link to broader number-theoretic frameworks. The parametric framework provides infinite families of solutions by exploiting algebraic identities and modular arithmetic. Future work could explore non-parametric solutions or generalizations to higher exponents.

Integer design of solutions Diophantine Equation $\boldsymbol{\alpha}(X^4+Y^4)(p^2+q^2+r^2+s^2) = (t^2+u^2)(C^2-D^2)(Z^2-W^2)P^{\beta}$ with $\alpha > 0, \beta > 0$ with Mathematical induction method for $\beta = 1,2,3,4, \dots$ *and so on*. Having integer design of solutions for $\beta > 2$ is

x $= k^n, y = k^{n+1}, z = k^{n+3}, w = k^{n+2}, p = k^n$, $C = \left(\left(\frac{1+k^4}{2}\right)^2 + 1\right)n$,

$D = \left(\left(\frac{1+k^4}{2}\right)^2 - 1\right)n, \alpha = (1+k^4)(k^6-k^4)k^{(\beta-2)n}n^2, q = p+1,\ r = \frac{p^2-1}{2}$,

$s = \left(\frac{p+1}{2}\right)^2 - 1$, $t = \frac{p^2+1}{2}$, $u = \left(\frac{p+1}{2}\right)^2 + 1$ .whenever $1+k^4$ is even, p is odd

and

x $= k^n, y = k^{n+1}, z = k^{n+3}, w = k^{n+2}, p = k^n$, $C = \left(\frac{(1+k^4)^2+1}{2})\right)n, D = \left(\frac{(1+k^4)^2-1}{2}\right)n,\ \alpha = (1+k^4)(k^6-k^4)k^{(\beta-2)n}n^2, q = p+1,\ r = \frac{p^2-1}{2}$, $s = \left(\frac{p+1}{2}\right)^2 - 1$, $t = \frac{p^2+1}{2}$,

$u = \left(\frac{p+1}{2}\right)^2 + 1$. if $1+k^4$ and p is odd. Here k must be positive integer

References:

[1]. Sridevi, K., & **Srinivas, T.** (2023). Transcendental representation of Diophantine equation and some of its inherent properties. *Materials Today: Proceedings*, *80*, 1822-1825.

[2]. Sridevi, K., & **Srinivas, T.** (2023). Existence Of Inner Addition and Inner Multiplication On Set of Triangular Numbers and Some Inherent properties of Triangular Numbers. *Materials Today: Proceedings*, *80*, 1822-1825.

[3]. Sridevi, K., & **Srinivas, T.** (2023). Cryptographic coding To Define Binary Operation on Set of Pythagorean Triples. *Materials Today: Proceedings*, *80*, 1822-1825.

[4]. Srinivas, T., & Sridevi, K. (2022). Transcendental representation of Diophantine Equation $x^n + y^n = z^n$ to Generate At most All Pythagorean and Reciprocal Pythagorean Triples. *JOURNAL OF ALGEBRAIC STATISTICS*, *13*(2), 3600-3609.

[5]. Sridevi, K., & **Srinivas, T**. (2022). Algebraic Structure Of Reciprocal Pythagorean Triples. Advances And Applications In Mathematical Sciences, Volume 21, Issue 3, January 2022, P.1315-1327© 2022 Mili Publications, India 0974-6803.

[6]. Srinivas, T., & Sridevi, K. (2022, May). A New Approach to Define Length of Pythagorean Triples and Geometric Series Representation of Set of Pythagorean Triples. In *Journal of Physics: Conference Series* (Vol. 2267, No. 1, p. 012059).ISSN:1742-6596. IOP Publishing.

[7]. Srinivas, T., & Sridevi, K. (2022, January). A new approach to define a new integer sequences of Fibonacci type numbers with using of third order linear Recurrence relations. In *AIP Conference Proceedings* (Vol. 2385, No. 1, p. 130005).ISSN 1551-7616.

[8]. Srinivas, T., & Sridevi, K. (2021, November). A New approach to define Algebraic Structure and Some Homomorphism Functions on Set of Pythagorean Triples and Set of Reciprocal Pythagorean Triples " in JSR (Journal of Scientific Research) , Volume 65, Issue 9, November 2021, Pages 86-92.ISSN: 0447-9483.

[9]. Sridevi, K., & Srinivas, T. (2020). A new approach to define two types of binary operations on set of Pythagorean triples to form as at most commutative cyclic semi group. *Journal of Critical Reviews*, *7*(19), 9871-9878.

[10]. SRINIVAS,. T. (2020). Proof Of Fermat's Last Theorem By Choosing Two Unknowns in the Integer Solution Are Prime Exponents. *pacific international Journal*, *3*(4), 147-151. ISSN 2616-4825 [Online] 2663-8991 [Print] Volume number 03 , issue number 03

[11]. Srinivas, T. (2014). Transportation and its health implications in India. *Int. J. Eng. Res. Dev*, *10*(7), 67-73. ISSN 2349-6185

[12]. Baggidi, P. R., Prasad, D. G., & **Srinivas, T**. (2013). Security enhancement in mobile ad hoc networks using non-zero noncooperative game theory. *International Journal of Research in Computer and Communication Technology*, *2*(8), 614-621.ISSN:2278-5841.

[13]. Srinivas, T., & Bathul) S,(2013). Monotonicity , IJSR (Issn :2277-8179)

[14]. Srinivas, T., & Bathul S,(2013). Approximate Derivatives , IJSR (Issn :2277-8179)

[15]. Srinivas, T., & Devi(2016) P. S. The Conventional Method of Optimizing Delivery Routes And Schedules.IJRCCT(2320-5156)

[16]. Srinivas, T., & Devi,(2017) P. S Effective Ness And Illustration Of Distribution Systems And Supply Chain Management.IJECS(Issn:2319-7242))

[17]. Srinivas T(2018), Taguchi Philosophy of Engineering Quality" in Pacific International Journal, ISSN 2616-4825 [Online] 2663-8991 [Print] Volume number 03 , issue number 03

[18]. Srinivas T(2018), Financial Soundness Of Selected Indian Petroleum Companies Using Altman Z–Score Model in Pacific International Journal, ISSN 2616-4825 [Online] 2663-8991 [Print] Volume number 03 , issue number 03.

[19]. Srinivas, T., & Ashok Kumar. C (2024). Construction of Pythagorean and Reciprocal Pythagorean n-tuples in Accelerating Discoveries in Data Science and Artificial Intelligence II, Springer Proceedings in Mathematics & Statistics 438, https://doi.org/10.1007/978-3-031-51163-9_4

[20]. Srinivas, T., & Sridevi, K(2024). A new approach to determine constant coefficients in higher order linear recurrence relations and repeated steps of their residues with mth integer modulo of some Fibonacci type numbers in 3rd International Conference on Functional Materials, Manufacturing, and Performances, AIP Conf. Proc. 2986, 030177-1–030177-9; https://doi.org/10.1063/5.0192504

[21]. Srinivas T(2023), Symmetric Key generation And Tree Construction in Cryptosystem based on Pythagorean and Reciprocal Pythagorean Triples in QEIOS, https://doi.org/10.32388/MTTDWD

[22]. Srinivas T(2024), Additive and Multiplicative Operations on Set of Polygonal Numbers in QEIOS https://doi.org/10.32388/MY0OLE

[23].Srinivas T(2023), Some Inherent Properties of Pythagorean Triples in *Research Highlights in Mathematics and Computer Science Vol. 7*, 18 March 2023 , Page 156-169 https://doi.org/10.9734/bpi/rhmcs/v7/18767D **Published:** 2023-03-18.

[24] Srinivas, T. (2024).Construction of Pythagorean and Reciprocal Pythagorean n-tuples. Springer Proceedings in Mathematics and Statistics.

[25] Srinivas, T. (2025). a book of Compact Mathematics for Undergraduate: Formulas & Identities-part I. BP international Publishers.

[26]. Srinivas,T.(2025). A Study on Integer Design of Exponential Solutions of Diophantine Equations $\boldsymbol{\alpha}(\boldsymbol{X}^4+\boldsymbol{Y}^4)^2=(\boldsymbol{C}^2+\boldsymbol{D}^2)(\boldsymbol{Z}^2+\boldsymbol{W}^2)^2\boldsymbol{P}^{\boldsymbol{\beta}}$ With $\boldsymbol{\alpha}>\mathbf{0}$, $\boldsymbol{\beta}=\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},5,6,7$ and $\boldsymbol{x}<\boldsymbol{y}<\boldsymbol{w}<\boldsymbol{z}$.

International Journal of Advanced Research in Science, Engineering and Technology Vol. 12, Issue 10, October 2025.

[27]. Srinivas,T.(2025). A Study on Integer Design of Exponential Solutions of Diophantine Equations $\alpha(X^4+Y^4)^2=(C^2+D^2)(Z^2-W^2)^2P^\beta$ With $\alpha>0$, $\beta=1,2,3,4$,5,6,7 and $x<y<w<z$. International Journal of Advanced Research in Science, Engineering and Technology Vol. 12, Issue 10, October 2025.

[28]. Srinivas,T.(2025). A Study on Integer Design of Exponential Solutions of Diophantine Equations $\alpha(X^4+Y^4)^2=(C^2+D^2)(Z^2+W^2)\ P^\beta$ With $\alpha>0$, $\beta=1,2,3,4$,5,6,7 and $x<y<w<z$. International Journal of Advanced Research in Science, Engineering and Technology Vol. 12, Issue 10, October 2025.

[29]. Srinivas,T.(2025). **A Study On Integer Design Of Solutions Of Diophantine Equation $\alpha(X^4+Y^4)^2(2U^2+V^2)=T^2(C^2+D^2)(Z^2-W^2)P^\beta$ With $\alpha>0,\beta=1,2,3,4,5,6,7$ *and* $x<y<w<z$, 2025 IJCRT | Volume 13, Issue 11 November 2025 | ISSN: 2320-2882.** [30].

[30]. Srinivas,T.(2025). **A Study On Integer Design Of Solutions Of Diophantine Equation $\alpha(X^4+Y^4)^2(2U^2+V^2)=T^2(C^2+D^2)(Z^2+W^2)P^\beta$ With $\alpha>0,\beta=1,2,3,4,5,6,7$ *and* $x<y<w<z$, 2025 IJCRT | Volume 13, Issue 11 November 2025 | ISSN: 2320-28**

[31]. Srinivas,T.(2025). **A Study On Integer Design Of Solutions Of Diophantine Equation $\alpha(X^4+Y^4)^2(3U^2+V^2)=T^2(C^2+D^2)(Z^2+W^2)P^\beta$ With $\alpha>0,\beta=1,2,3,4,5,6,7$ *and* $x<y<w<z$, 2025 IJCRT | Volume 13, Issue 11 November 2025 | ISSN: 2320-28**

[32]. Srinivas,T.(2025). **A Study On Integer Design Of Solutions Of Diophantine Equation $\alpha(X^4+Y^4)^2(2U^2+V^2)=T^2(C^2+D^2)(Z^2-W^2)P^\beta$ With $\alpha>0,\beta=1,2,3,4,5,6,7$ *and* $x<y<w<z$, 2025 IJCRT | Volume 13, Issue 11 November 2025 | ISSN: 2320-28.**

[33]. Srinivas,T.(2025). **A Study On Integer Design Of Solutions Of Diophantine Equation $\alpha(X^4+Y^4)(\gamma U^2+V^2)=T^2(C^2+D^2)(Z^2-W^2)P^\beta$ With α, $\beta>0$, $\gamma=2,3$ *and* $x<y<w<z$, 2025 IJCRT | Volume 13, Issue 11 November 2025 | ISSN: 2320-28**

[34]. Srinivas,T.,G.Sujatha(2025). **Solving For Stoichiometric Coefficients Of Chemical Diophantine Equation** $\alpha(X^4+Y^4)^2(5U^2+V^2)=T^2(C^2+D^2)(Z^2\text{-}W^2)P^{\beta}$ **With** $\alpha>0,\beta=1,2,3,4,5,6,7$ and $x<y<w<z$, **2025 IJCRT | Volume 13, Issue 11 November 2025 | ISSN: 2320-28**

[35]. Srinivas,T.,G.Sujatha(2025), SOLVING FOR STOICHIOMETRIC OEFFICIENTS OF CHEMICAL DIOPHANTINE EQUATION $\alpha(X^4+Y^4)^2(5U^2+V^2)=T^2(C^2+D^2)(Z^2\text{-}W^2)P^{\beta}$ WITH **α>0,β=1,2,3,4,5,6,7 and x<y<w<z.** International Research Journal of Modernization in Engineering Technology and Science **Volume:07/Issue:11/November-2025.**

[36]. Srinivas,T.,G.Sujatha(2025), **Integer Design Of Solutions Of One Of The Complex Stoichiometric Reaction System** $\alpha(X^4+Y^4)^2(21U^2+V^2)=T^2(C^2+D^2)(Z^2\text{-}W^2)P^{\beta}$ WITH **α>0,β=1,2,3,4,5,6,7 and x<y<w<z.** YMER || ISSN : 0044-0477