Stability Properties of the Boussinesq Equations on an Infinite Strip

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Abstract

In this paper, we consider a smooth equilibrium solution (u_0, p_0, τ_0) of the Boussinesq Equations on an infinite strip($\Omega = \mathbb{R} \times]-\frac{1}{2},\frac{1}{2}[$).

We examine the analysis presented in [3] to investigate whether similar simplifications to those found in [8] and [1] can be achieved by reducing the spatial domain from the infinite plate $\Omega = \mathbb{R}^2 \times]-\frac{1}{2},\frac{1}{2}[$ to the infinite strip $\Omega = \mathbb{R} \times]-\frac{1}{2},\frac{1}{2}[$.

Keywords: *Boussinesq equation, Bénard equations, Navier-Stokes equations, Ljapunov stability, Ljapunov instability, Regularity.*

1. Notations

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For X, Y Banach spaces, $\| . \|_X$, $\| . \|_Y$ are their respective norms. $L(X, Y)$ is the space of bounded operators from X to Y with $\| T \|$ the operator norm.

For A a linear operator on X and $E \subseteq X$ a subspace, $A |_E$ is the restriction of A to E.

For any Ω, $H^P(\Omega)$ is the Sobolev space of functions having square integrable derivatives up to order p with $(.,.)_p$ and $\| . \|_{H^P(\Omega)}$ the usual scalar product and norm on $H^P(\Omega)$. We set $\mathcal{L}^2(\Omega) = H^0(\Omega)$ and $\| . \|_{H^P} = \| . \|_{H^P(\Omega)}$ and extend this notation to vectors and set :

$$\| u \|_{\mathcal{L}^2}^2 = \| u_1 \|_{\mathcal{L}^2}^2 + \| u_2 \|_{\mathcal{L}^2}^2 + \| u_3 \|_{\mathcal{L}^2}^2 \tag{1.1}$$

where $u = (u_1 , u_2, u_3) \in \left(\mathcal{L}^2(\Omega)\right)^3$, Likewise with the Sobolev norms. The scalar product on $(H^P(\Omega))^3$ is $\langle .,. \rangle_p$, with :

$$\langle u, v \rangle_p = \sum_{i=1}^{3} (u_i, v_i)_p , \qquad u_i, v_i \in H^p(\Omega) \tag{1.2}$$

where $u = (u_1, u_2, u_3), ; v = (v_1, v_2, v_3)$ we set $\langle .,. \rangle = \langle .,. \rangle_0$.

$C^p(\overline{\Omega})$ is the space of functions p times continously differentiable on $\overline{\Omega}$ and $C_0^p(\overline{\Omega})$ is the space of functions $f \in C^p\left(\overline{\Omega}\right)$ with $supp(f)$ compact.

2. Introduction

The Boussinesq equations describe the flow of an incompressible fluid under buoyancy forces due to temperature differences. In this simplified setup, we consider a $2D$ infinite strip with horizontal periodicity and vertical boundary conditions. Our purpose is to determine the stability of the equilibrium solution.

For this we investigate the stability of equilibrium solutions to the Boussinesq equations in an infinite strip heated from below. The equilibrium solutions are assumed to be spatially periodic in the x −direction. While the periodic case has been well-studied, the stability analysis differs significantly for non-periodic perturbations. We primarily analyze disturbances belonging to the space $\mathcal{L}^2(\Omega)$ in the x −direction. We systematically compare the stability properties under both periodic and non-periodic ($\mathcal{L}^2$) perturbations, characterizing their fundamental relationship.

In this paper, we study the stability of certain equilibrium solutions of the Bénard equations given by:

$$\begin{cases} \partial_t u = \nu\Delta u - \nabla p - g\big(1-\alpha(T-t_0)\big)j - (u.\nabla)\mathrm{u}, \\ \partial_t T = \kappa\Delta T - (u.\nabla)T, \qquad\qquad with\, \nabla.\mathrm{u} = 0 \end{cases} \tag{2.1}$$

on the infinite strip domain $\Omega = \mathbb{R}\times]-\frac{1}{2},\frac{1}{2}[$. Here, u denotes the velocity field, p the pressure, and T the temperature. The constants are as follows: g is the gravitational acceleration, $\alpha > 0$ is the thermal expansion coefficient, and $j = (0, 1)$ is the unit vector in the vertical direction. The parameters ν and κ represent the kinematic viscosity and thermal conductivity, respectively.

We impose Dirichlet boundary conditions on the velocity field: $u = 0$ on $\partial\Omega$. For the temperature, we assume fixed boundary values: $T\left(x,-\frac{1}{2}\right) = t_0$ and $T\left(x,\frac{1}{2}\right) = t_1$, where $t_0 > t_1$.

Our focus is on equilibrium solutions (u_0, τ_0, p_0) of system (2.1) that are sufficiently smooth on Ω, satisfy the given boundary conditions, and are periodic in the horizontal direction x with period L, a property we refer to as L −periodicity.

We investigate the stability of such equilibria from two perspectives:

1. Under small perturbations $u - u_0, T - \tau_0, p - p_0$ that are also L −periodic and satisfy the same boundary conditions.
2. Under general perturbations u,T,p which satisfy the same boundary conditions, but for which the differences satisfy:

$$u - u_0, \nabla(p - p_0) \in \left(\mathcal{L}^2(\Omega)\right)^2, and\; T - \tau_0 \in \mathcal{L}^2(\Omega).$$

3. Preliminaries and Main Results

We now summarize key results from [1, 8] that will be essential for our subsequent analysis.

Consider an arbitrary but fixed L −periodic equilibrium solution (u_0, τ_0, p_0) of system (2.1), as introduced previously. To examine the stability of this solution under both L −periodic and $\mathcal{L}^2(\Omega)$ perturbations, we introduce the substitutions:

$$u = u_0 + v,$$
$$T = \tau_0 + \vartheta,$$

$$p = p_0 + \pi.$$

After straightforward (non-scaled) computations, these substitutions yield the following system of equations:

$$\begin{cases}\partial_t v = \nu\Delta v - (u_0.\nabla)v - (v.\nabla)u_0 - \nabla\pi - \alpha g\vartheta j,\\ \partial_t T = \kappa\Delta\vartheta - (u_0.\nabla)\vartheta - (v.\nabla)\tau_0 - (v.\nabla)\vartheta\end{cases} \quad (3.1)$$

Subject to the conditions $\nabla\vartheta = 0$ and $v = \vartheta = 0$ on $\partial\Omega$, equation (3.1) requires an appropriate functional framework, depending on the type of stability under consideration.

In the case of $\mathcal{L}^2(\Omega)$ −stability, the relevant concepts are as follows:

We define the space $E \subseteq (\mathcal{L}^2(\Omega))^2$ of divergence-free vector fields as the closure in $(\mathcal{L}^2(\Omega))^2$ of the set of smooth, compactly supported vector fields that are divergence-free, i.e.,

$$E = closure\ in\ (\mathcal{L}^2(\Omega))^2\ of\ all\ f \in H_0^1(\Omega)^2\ such\ that\ \nabla \cdot f = 0. \quad (3.2)$$

We denote by P be the orthogonal projection from $(\mathcal{L}^2(\Omega))^2$ onto the subspace E and define the extended projection $\mathcal{P}$ on $(\mathcal{L}^2(\Omega))^3$ by:

$$\mathcal{P}(v,\vartheta) = (Pv,\vartheta);\ \ v \in \left(\mathcal{L}^2(\Omega)\right)^2, \vartheta \in \mathcal{L}^2(\Omega) \quad (3.3)$$

Where (v,ϑ) represents the triplet (v_1, v_2, ϑ) for $v = (v_1, v_2)$. The Stokes operator A_S on E is defined by

$$\begin{gathered} v \in \text{dom}(A_s) \text{ iff } v \in (H^2(\Omega) \cap H_0^1(\Omega))^2 \text{and} \\ \text{div } v = 0 \text{ and } A_S v = \nu P\Delta v \text{ for such } v \end{gathered} \quad (3.4)$$

Finally, we define an operator B_S on the product space $E \times \mathcal{L}^2(\Omega)$ in terms of A_S

$$\begin{gathered} (v,\vartheta) \in \text{dom}(B_S) \text{ iff } v \in \text{dom}(A_s), \vartheta \in H^2(\Omega) \cap H_0^1(\Omega) \\ \text{and } B_S(v,\vartheta) = (A_S v, \kappa\Delta\vartheta) \text{ for such } (v,\vartheta) \end{gathered} \quad (3.5)$$

The properties of A_S are thoroughly examined in [1],[8], with their extension to B_S developed in [6], Section VIII. These analyses reveal that both A_S and B_S are self-adjoint operators on their respective domains E and $E \times \mathcal{L}^2(\Omega)$. Furthermore, they satisfy the conditions $A_S \le -\epsilon$ and $B_S \le -\epsilon$ for some $\epsilon > 0$. To complete our formulation, we must incorporate the additional linear terms from (3.1), which we accomplish by defining:

$$\begin{gathered} T_0 v = -(v.\nabla)u_0 - (u_0.\nabla)v \\ T_1(v,\vartheta) = (T_0 v + \alpha g\vartheta j, -(u_0.\nabla)\vartheta - (v.\nabla)u_0 \end{gathered} \quad (3.6)$$

If we supply (3.6) with the stipulation:

$$\text{dom}(T_0) = H^1(\Omega)^2,\quad \text{dom}(T_1) = H^1(\Omega)^2 \quad (3.7)$$

And N is no-linear operator defined on E by $N(v,\vartheta) = (0, -v\nabla\vartheta)$.

Under these conditions T_1 becomes an operator that is bounded relative to B_S ([9], pg. 190, [4] , [8], [6] sect. VIII). Consequently, the operator $B_S + \mathcal{P}T_1$ generates a holomorphic semigroup on $E \times \mathcal{L}^2(\Omega)$. The main result derived in [8] on the basis of [4] , [1] and [5], establishes:

(a) If $\sigma(B_S + \mathcal{P}T_1) \subseteq \{\lambda/\ Re(\lambda) \le -\epsilon\}$ for some $\epsilon > 0$, then the equilibrium solution u_0, τ_0, p_0 is Ljapounov stable;

(b) If there exists $\lambda \in \sigma(BS + \mathcal{P}T_1)$ with $Re(\lambda) > 0$, then u_0, τ_0, p_0 is Ljapounov unstable.

Following the main theorem in [8] and its extension to the Bénard case in [6], we can characterize $\sigma(B_S + \mathcal{P}T_1)$ through $\Theta -$ periodic spectra, which enables computational approaches in certain scenarios.

In order to explain Θ −periodicity we fix $\epsilon > 0$ small and set $M_\epsilon =]-\epsilon, 2\pi + \epsilon[$, $M = [0,2\pi]$ and let $\dot{M}_\epsilon = M_\epsilon - \{0,2\pi\} =]0,2\pi[$.

With $Q_L =]0, L[$ we set $Q = Q_L \times \mathbb{R}$. For $\Theta \in M_\epsilon$ we say $f \in C^p_\Theta(Q)$ iff $f \in C^p(\overline{\Omega})$ and satisfies the Θ −periodicity condition:

$$\begin{aligned} &f(x+kL, y) = e^{ik\Theta} f(x,y), k \in \mathbb{Z} \\ &\text{for } (x,y) \in \overline{\Omega} = \mathbb{R} \times [-\tfrac{1}{2}, \tfrac{1}{2}] \end{aligned} \tag{3.8}$$

Sobolev spaces are introduced in a standard manner: a function $f \in H^p_\Theta(Q)$ iff $f \in \mathcal{L}^2(\Omega)$ and there exists a sequence $(f_n)_{n\in\mathbb{N}} \in C^p_\Theta(Q)$ such that $\lim \left|\left|f - fn\right|\right|_{H^p} = 0$ where $(H^p = H^p(Q)$. Similarly, a function $f \in C^1_{\Theta,0}(Q)$ iff $f \in C^1_\Theta(Q)$ and $\left(x, \pm\frac{1}{2}\right) = 0$ for $x \in \mathbb{R}$. Then $f \in H^p_{\Theta,0}(Q)$ iff $\lim\left|\left|f - fn\right|\right|_{H^1} = 0$ for some sequence $(f_n)_{n\in\mathbb{N}} \in C^p_{\Theta,0}(Q)$ (see [1] for details). One now defines Θ −periodic counterparts of the objects $E, P, \mathcal{P}, A_S, B_S$. Specifically, we define:

$$E_\Theta \text{ is the } \mathcal{L}^2(Q) - \text{Closure of the set } f \in H^1_{\Theta,0}(Q)^2 \text{ such that } \nabla. f = 0 \tag{3.9}$$

P_Θ is the orthogonal projection from $\mathcal{L}^2(Q)^2$onto E_Θ while $\mathcal{P}_\Theta$ is given by

$$\mathcal{P}_\Theta(v,\vartheta) = (P_\Theta v, \vartheta), v \in \mathcal{L}^2(Q)^2, \vartheta \in \mathcal{L}^2(Q) \tag{3.10}$$

The Θ −periodic Stokes operator $A_S(\Theta)$ is defined as follows:

$$\begin{aligned} &v \in \operatorname{dom}\big(A_s(\Theta)\big) \text{ iff } v \in (H^2_\Theta(\mathrm{Q}) \cap H^1_{\Theta,0}(\mathrm{Q}))^2, div\ v = 0 \\ &\text{and } A_S(\Theta)v = \nu P_\Theta \Delta v \text{ for such } v \end{aligned} \tag{3.11}$$

The operator $B_S(\Theta)$ is defined in terms of $A_S(\Theta)$ by:

$$\begin{aligned} &(v,\vartheta) \in \operatorname{dom}(B_S(\Theta)) \text{ iff } v \in \operatorname{dom}(A_s(\Theta)), \vartheta \in H^2_\Theta(\mathrm{Q}) \cap H^1_{\Theta,0}(\mathrm{Q}) \\ &\text{and } B_S(\Theta)(v,\vartheta) = (A_S(\Theta)v, \kappa\Delta\vartheta) \text{ in this case.} \end{aligned} \tag{3.12}$$

The fundamental properties of $A_S(\Theta), B_S(\Theta)$ are thoroughly discussed in [6], [4]. In particular, both operators are selfadjoint on their respective domains $E_\Theta, E_\Theta \times \mathcal{L}^2(Q)$, and $\leq -\mu$ for some $\mu > 0$ independent of $\Theta \in M_\epsilon$. Moreover, they have compact resolvents. The operators T_0, T_1, defined formally by (3.6) acquire a precise meaning by setting:

$$\operatorname{dom}(T_0) = H^1(Q)^2, \quad \operatorname{dom}(T_1) = H^1(Q)^3 \tag{3.13}$$

Based on [4] it is shown in [6], that T_1 is relatively bounded with respect to $B_S(\Theta)$. As a result, the operator $B_S(\Theta) + \mathcal{P}_\Theta T_1$ generates a holomorphic semigroup on $E_\Theta \times \mathcal{L}_2(Q)$, and it also has compact resolvent.

The periodic case corresponds to $\Theta = 0$ or $\Theta = 2\pi$. to emphasize this situation, one writes $H^p_{per}(Q), H^p_{per,1}(Q), B_s(per)$ etc. instead of $H^p_0(Q), H^p_{0,1}(Q), B_s(0)$ respectively. Although the definitions remain unchanged for $\Theta \in \dot{M}_\epsilon$, the periodic case can be viewed as is the limit of the Θ −periodic case as $\Theta \to 0, \Theta \in \dot{M}_\epsilon$. A more refined analysis is therefore necessary, leading to the introduction of a subclass of L −periodic vector fields.

Proposition 3.1. B_S is selfadjoint and $\leq -\epsilon$ for some $\epsilon > 0$, and exists $C > 0$ such if $B_S(v,\vartheta) = f$ for $(v,\vartheta) \in D(B_S)$ and $f \in E$. then $\left|\left|(v,\vartheta)\right|\right|_{H^2(\Omega)} \leq C\left|\left|f\right|\right|_{\mathcal{L}^2(\Omega)^3}$

Proof. As indicated in [8] and [1] there exist constants $\epsilon_1, \epsilon_2 > 0$ such that The Stokes operator A_s and the Laplace operator Δ are selfadjoints and satisfy $\leq -\epsilon 1$ and $\leq -\epsilon 2$ respectively.

Furthermore, there exists $C_1, C_2 > 0$ such that if $A_S v = f_1$ and $\kappa\Delta\vartheta = f_2$ for $(v, \vartheta) \in D(B_S)$ and $f = (f_1, f_2) \in E$, then setting $C = max\{C_1, C_2\}$, $\epsilon = min\{\epsilon_1, \epsilon_2\}$, the operator B_S is selfadjoint and $\leq -\epsilon$, and $\left|\left|v, \vartheta\right|\right|_{H^2(\Omega)} \leq C\left|\left|f\right|\right|_{\mathcal{L}^2(\Omega)^3}$ ■

Remark 3.2. According to the inequality in Proposition 3.1, we conclude that the singularity of the operator B_s vanishes at $\Theta = 0$ and $\Theta = 2\pi$, as in the case of the Navier–Stokes equations in an infinite strip [1] .

Corollary 3.3. T_1 is relatively bounded with respect to B_S ie : for all $\delta > 0$ there is $K_\delta > 0$ such that: $\| T_1 u \|_{\mathcal{L}^2(\Omega)^3} \leq \delta \| B_S u \|_{\mathcal{L}^2(\Omega)^3} + K_\delta \| u \|_{\mathcal{L}^2(\Omega)^3}$, for all $u \in D(B_S)$.

Proof. Let A be an operator on $\mathcal{L}^2(\Omega)^3$ defined by

$$(Au)_i = \Delta u_i, \forall u \in D(A), D(A) = \left(H^2(\Omega) \cap H_0^1(\Omega)\right)^3, \mathrm{i} = 1{,}2{,}3.$$

Assume that Ω is bounded in the $\vec{y}$ −direction. Then, by the Poincaré inequality, the operator A is bounded in the sense that there exists a constant $c' > 0$ such that

$$\| Au \|_{\mathcal{L}^2(\Omega)^3} \leq c' \| u \|_{H^2(\Omega)^3} \ \forall u \in D(A).$$

Moreover, since $A \leq -\varepsilon$ for some $\varepsilon > 0$, the fractional power $(-A)^{1/2}$ is well-defined and bounded. We admit for the moment the following intermediate result: for any $\delta > 0$, there exists a constant $K_\delta > 0$ such that

$$\| (-A)^{\frac{1}{2}} u \|_{\mathcal{L}^2(\Omega)^3} \leq \delta \| Au \|_{\mathcal{L}^2(\Omega)^3} + k_\delta \| u \|_{\mathcal{L}^2(\Omega)^3} \ \forall u \in D((-A)^{\frac{1}{2}}). \tag{3.14}$$

Furthermore, for all $u \in D\left((-A)^{\frac{1}{2}}\right) = (H_0^1(\Omega))^3$, it holds that

$$\| (-A)^{\frac{1}{2}} u \|_{\mathcal{L}^2(\Omega)^3} = (\sum_{i=1}^{3} (\nabla u_i, \nabla u_i))^{1/2}$$

According to Proposition 3.1, there exists a constant $c > 0$ such that

$$\| u \|_{H^2(\Omega)^3} \leq c \| B_s u \|_{\mathcal{L}^2(\Omega)^3}$$

Now, for all $u \in (H_0^1(\Omega))^3$, we have the estimate

$$\begin{aligned} \| T_1 u \|_{\mathcal{L}^2(\Omega)^3} &\leq c''(\textstyle\sum_{i=1}^3 (\nabla u_i, \nabla u_i))^{\frac{1}{2}} + \| u \|_{\mathcal{L}^2(\Omega)^3} \\ &= c'' \| (-A)^{\frac{1}{2}} u \|_{\mathcal{L}^2(\Omega)^3} + c'' \| u \|_{\mathcal{L}^2(\Omega)^3} \end{aligned}$$

Applying inequality (3.14), we obtain

$$\| T_1 u \|_{\mathcal{L}^2(\Omega)^3} \leq c(\delta \| \mathrm{Au} \|_{\mathcal{L}^2(\Omega)^3} + \mathrm{k}_\delta \| \mathrm{u} \|_{\mathcal{L}^2(\Omega)^3}) + \mathrm{c} \| \mathrm{u} \|_{\mathcal{L}^2(\Omega)^3}$$

$$= \quad c''\delta \parallel Au \parallel_{\mathcal{L}^2(\Omega)^3} + c''(k_\delta + 1) \parallel u \parallel_{\mathcal{L}^2(\Omega)^3}$$

From the inequality $\parallel Au \parallel_{\mathcal{L}^2(\Omega)^3} \leq c' \parallel u \parallel_{H^2(\Omega)^3}$ we can deduce that

$$\parallel T_1 u \parallel_{\mathcal{L}^2(\Omega)^3} \leq c''\delta c' \parallel u \parallel_{H^2(\Omega)^3} + c''(k_\delta + 1) \parallel u \parallel_{\mathcal{L}^2(\Omega)^3}$$

Finally, using the regularity estimate $\parallel u \parallel_{H^2(\Omega)^3} \leq c \parallel B_s u \parallel_{\mathcal{L}^2(\Omega)^3}$, it follows that

$$\parallel T_1 u \parallel_{\mathcal{L}^2(\Omega)^3} \leq c''\delta c' c \parallel B_s u \parallel_{\mathcal{L}^2(\Omega)^3} + c''(k_\delta + 1) \parallel u \parallel_{\mathcal{L}^2(\Omega)^3}$$

■

Proof of the inequality (3.14)

Proof. We begin by recalling the elementary inequality: for all $\epsilon > 0$ and all $\lambda \geq 0$,

$$\lambda \leq \epsilon\lambda^2 + \frac{1}{4\epsilon} \qquad (a)$$

Now, consider the operator A defined on the domain

$$D(A) = \left(H^2(\Omega) \cap H_0^1(\Omega)\right)^3$$

With action

$$Au = -I\Delta u = (-\Delta u_1, -\Delta u_2, -\Delta u_3), \text{for } u = (u_1, u_2, u_3) \in D(A)$$

Where I is the 3×3 identity matrix.

According to Proposition 3.1, the operator A is self-adjoint and satisfies

$$A \geq \delta I, \quad For\ some\ \delta > 0.$$

Let $(E_\lambda)_{\lambda \geq \delta}$ denote the spectral family associated with A. By the spectral theorem,

$$A = \int_\delta^{+\infty} \lambda dE_\lambda, \qquad (3.15)$$

$$A^{\frac{1}{2}} = \int_\delta^{+\infty} \lambda^{\frac{1}{2}} dE_\lambda, \qquad (b)$$

Using identities (a) and (b), and by straightforward calculations, we obtain for all $u \in D(A)$:

$$\parallel A^{\frac{1}{2}} u \parallel^2_{\mathcal{L}^2(\Omega)^3} \leq \epsilon \parallel Au \parallel^2_{\mathcal{L}^2(\Omega)^3} + \frac{1}{4\epsilon} \parallel u \parallel^2_{\mathcal{L}^2(\Omega)^3}$$

Hence,

$$\| A^{\frac{1}{2}}u \|_{\mathcal{L}^2(\Omega)^3} \leq \sqrt{\epsilon} \| Au \|_{\mathcal{L}^2(\Omega)^3} + \frac{1}{2\sqrt{\epsilon}} \| u \|_{\mathcal{L}^2(\Omega)^3}, \forall u \in D(A) \qquad (c)$$

We also recall that there exists a constant $c_0 > 0$ such that

$$\| Au \|^2_{\mathcal{L}^2(\Omega)^3} \leq c_0 \| u \|_{H^2(\Omega)^3}, \quad \forall u \in D(A) \qquad (d)$$

Moreover, from the theory of quadratic forms (see [7]), it is known that

$$D\left(A^{\frac{1}{2}}\right) = \left(H_0^1(\Omega)\right)^3 \text{ and } \| A^{\frac{1}{2}}u \|^2_{\mathcal{L}^2(\Omega)^3} = \sum_{i=1}^{3} \| \nabla u_i \|^2_{\mathcal{L}^2(\Omega)^3} \qquad (e)$$

Now, for any $u = (u_1, u_2, u_3) \in D(A)$, combining (c), (d) and Proposition 3.1, we obtain:

$$\sum_{i=1}^{3} \| \nabla u_i \|^2_{\mathcal{L}^2(\Omega)^3} \leq \epsilon c_0 c \| Au \|^2_{\mathcal{L}^2(\Omega)^3} + \frac{1}{4\epsilon} \| u \|^2_{\mathcal{L}^2(\Omega)^3} \qquad (3.16)$$

where c is the constant from Proposition 3.1.

It is straightforward to verify that

$$\| Au \|^2_{\mathcal{L}^2(\Omega)^3} \leq c_0 \| u \|^2_{H^2(\Omega)^3}, \text{for all } u \in D(A) \qquad (3.17)$$

for some constant $c_0 > 0$.

Moreover, it is well known—see, for instance, the theory of quadratic forms as developed in [7]—that the domain of the square root of A satisfies

$$D\left(A^{\frac{1}{2}}\right) = \left(H_0^1(\Omega)\right)^3 \qquad (3.18)$$

and the associated norm is given by

$$\| A^{\frac{1}{2}}u \|^2_{\mathcal{L}^2(\Omega)^3} = \sum_{i=1}^{3} \| \nabla u_i \|^2_{\mathcal{L}^2(\Omega)^3} \qquad (3.19)$$

for any $u = (u_1, u_2; u_3) \in D(A^{\frac{1}{2}}) = H_0^1(\Omega)^3$

Now consider $u = (u_1, u_2; u_3) \in D(A)$. Combining (3.19) with estimate (3.17), and applying Proposition 3.1, we deduce the following inequality:

$$\sum_{i=1}^{3} \| \nabla u_i \|^2_{\mathcal{L}^2(\Omega)^3} \leq \epsilon c_0 c \| Au \|^2_{\mathcal{L}^2(\Omega)^3} + \frac{1}{4\epsilon} \| u \|^2_{\mathcal{L}^2(\Omega)^3} \qquad (3.20)$$

for any $\epsilon > 0$, where c is the constant appearing in Proposition 3.1. ■

Remark 3.4. According to Corollary 3.3, the operator T_1 is relatively bounded with respect to B_S. Consequently, the same holds for $\mathcal{P}T_1$. It then follows from the result in [2] that the operator $B_S + \mathcal{P}T_1$ generates a holomorphic semi-group on E.

Then, for each fixed $\lambda > 0$ such that $\lambda \in \rho(B_S + \mathcal{P}T_1)$ (the resolvent set of $B_S + \mathcal{P}T_1$), we can define the fractional powers of the operator $F_\lambda = \lambda - B_S + \mathcal{P}T_1$ by

$$F_\lambda^\gamma = (\lambda - B_S + \mathcal{P}T_1)^\gamma, \gamma > 0.$$

This allows us to introduce the corresponding fractional power spaces $X_\gamma = D(F_\lambda^\gamma)$,

Equipped with the norms

$$\| f \|_\gamma = \| F_\lambda^\gamma f \|_{\mathcal{L}^2(\Omega)^3}, f \in X_\gamma.$$

It is well known that the norms $\|\cdot\|_\gamma$ are equivalent for different choices of $\lambda > 0$, and the spaces X_γ are independent of λ in the sense of norm equivalence.

According to the results in [2], the non-linear operator N satisfies the following assertion:

for any $\gamma \in]\frac{3}{4}, 1[$ and any $v \in X_\gamma$, we have

$$N(v) \in \mathcal{L}^2(\Omega)^3 \text{ and } \| N(v) \|_{\mathcal{L}^2(\Omega)^3} \leq C_\gamma \|\cdot\|_\gamma^2$$

for some constant $C_\gamma > 0$ depending only on γ.

As a consequence, the original system (2.1) can be reformulated as an evolution equation in the sense of Pazy [2], where standard semi-group theory applies.

3. Comments

By adopting a methodology similar to that used in [8],[1] and rigorously following the steps developed in [3], we have recovered the results established for the Navier-Stokes equations in an infinite strip. This approach enabled us to generalise these results to the more complex framework of the Boussinesq equations in an infinite strip.

As noted earlier, the singularities at Θ = 0 and Θ = 2π are not arise in the computations

presented in the preceding sections. This exclusion leads to a simplification of the spectral theory developed for the three dimensional case ($d = 3$) in [3].

In light of this simplification, we restrict ourselves to a concise presentation of the resulting formulation.

Accordingly, the well-known spectral formula takes the following simplified form in our case:

$$\sigma(B_s + \mathcal{P}T_1) = \bigcup_{\Theta \in [0,2\pi]} \sigma(B_s(\Theta) + \mathcal{P}_\Theta T_1) \qquad (4.1)$$

From the spectral formula (4.1) and the equality $B_s(per) = B_s(0)$, we conclude:

$$B_s(per) + \mathcal{P}_{per}T_1 \subseteq B_s + \mathcal{P}T_1 \quad (4.2)$$

This inclusion ensures that Ljapunov stability with respect to perturbations in $\mathcal{L}^2$ implies Ljapunov stability with respect to L−periodic perturbations as well.

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