INTEGER DESIGN OF SOLUTIONS OF ONE OF THE COMPLEX STOICHIOMETRIC REACTION SYSTEMS $\alpha(X^4+Y^4)^2(21U^2+V^2)=T^2(C^2+D^2)(Z^2-W^2)P^\beta$ WITH $\alpha>0, \beta=1,2,3,4,5,6,7\ and\ x<y<w<z$

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Abstract:

Integer design of solutions in complex stoichiometric reaction systems involves systematically formulating and solving the atom-balance constraints as systems of integer linear equations, followed by computational approaches to identify minimal, feasible integer stoichiometric coefficients that satisfy conservation laws and additional constraints.

Diophantine equations play a fundamental role in bridging algebraic number theory with chemical analysis, particularly in representing and solving stoichiometric constraints in organic chemistry as systems of linear integer equations. This approach ensures that the total number of atoms of each element remains conserved on both sides of a chemical reaction, aligning with the principles of atomic and charge balance essential for accurate molecular modeling. For systems with more than eight unknowns or those involving higher degree relationships (exponential or nonlinear mixtures), solutions are derived using combinatorial and algebraic number theory techniques. The general structure is set up so that each linear or polynomial equation reflects conservation of one elemental or charge property, and the set of all equations defines the solution space of feasible integer stoichiometric coefficients. When the structure mimics classical forms (e.g., Pythagorean relations or exponential equations), parametrization methods from number theory are used:

from Reference [9], Applied the set of integer solutions $21U^2+V^2=T^2$, focused to find the general exponential integer solution of $\alpha\left(X^4+Y^4\right)^2\left(21U^2+V^2\right)=T^2\left(C^2+D^2\right)\left(Z^2-W^2\right)P^\beta$.

With $\alpha>0$, is derived from fixed value of $\beta=1,2,3,4,5,6,7\ and\ x<y<w<z$.

Keywords: Diophantine Equation, exponential, Pythagorean triplet, Integer design.

I. Introduction:

Diophantine equations, which are polynomial equations restricted to integer solutions, occupy a central place in algebraic number theory. Within this broad framework, exponential and higher-degree forms—including quintic Diophantine equations—extend their significance beyond pure mathematics into applied scientific domains. In particular, the field of organic chemistry frequently encounters problems demanding integer consistency and stoichiometric balance that naturally align with Diophantine principles.

Given empirical data such as molecular mass and elemental composition, integer solutions can be found for variables representing atom counts. These linear relationships between atomic numbers, governed by molecular constraints, define the integer framework underlying chemical structure analysis.

While most stoichiometric problems translate into linear Diophantine systems, certain advanced chemical analyses require nonlinear relationships between molecular components, leading to higher-degree or even quintic Diophantine equations. Such equations may emerge in theoretical modeling of complex organic networks, reaction kinetics, or molecular structure prediction. Although direct applications of quintic equations in routine stoichiometric balancing are rare, they establish a theoretical foundation for understanding non-linear dependencies within reaction mechanisms.

In this paper, we focus on solving systems representing stoichiometric constraints as Diophantine equations with more than eight unknowns. Specifically, we employ a methodological combination of mathematical induction, trial-and-error computation, and Pythagorean triplet generation to identify consistent integer solutions. This approach contributes to an emerging synthesis of algebraic number theory and organic chemical analysis, offering reliability, computational efficiency, and theoretical completeness in designing integer-based chemical models

In this paper, focused to find the general exponential integer solution of

The general exponential integer solution of $\boldsymbol{\alpha\left(X^4+Y^4\right)^2\left(21U^2+V^2\right)=T^2\left(C^2+D^2\right)\left(Z^2-W^2\right)P^{\beta}}$

With $\boldsymbol{\alpha>0}$**, is derived from fixed value of** $\boldsymbol{\beta=1,2,3,4,5,6,7\ and\ x<y<w<z.}$

Suppose in a hypothetical organic synthesis network, intermediates X, Y, U, V, C, D, Z, W, T, P represent integer counts of species or structural motifs constrained by nonlinear relations reflecting their combinatorial formation energies or symmetries. The equation guarantees that only sets of integers satisfying the polynomial equalities correspond to chemically valid configurations or reaction states.

For instance, if X, Y represent counts of two conjugated units raised to the fourth power to encode structural complexity, and C, D, Z, W relate to conserved quantities of molecular fragments or charge states, then this equation embodies a set of integer constraints that must be met simultaneously.

II. Literature Review:

The chemical reaction network or process is represented as a system of linear equations derived from the conservation of atoms and charge. The unknowns represent stoichiometric coefficients that must be integers. Techniques like Singular Value Decomposition (SVD) and Structured Target Factor Analysis (STFA) are employed to determine the minimum set of linearly independent stoichiometric reactions underlying the system. This reduces complexity by focusing on reaction bases rather than full linear combinations. Stoichiometric coefficients are constrained as positive integers (often less than 10 to remain chemically meaningful). Integer Linear Programming (ILP) or Mixed Integer Linear Programming (MILP) frameworks help find minimal, physically realistic coefficient sets satisfying atom-balance equations and additional constraints (e.g., monotonic reaction extents). Beyond stoichiometry, thermodynamic constraints (e.g., free energy changes) and kinetic behaviors are integrated or checked after stoichiometric design to ensure realizability. The stoichiometric model identifies feasible integer reaction setups independently of kinetic parameters, allowing modular design. This principle facilitates the optimal design of production reactors or reaction pathways in complex systems like pharmaceuticals. Modern methods rely on computational databases of reactions, iterative optimization, and cross-validation techniques to rank stoichiometric designs by predictive accuracy and practical efficiency.

III. Research Methodology, Results & Discussions:

Proportion 1: A Study on integer design of solution of above Diophantine Equation at

$\boldsymbol{\beta = 1}$ **is** $\boldsymbol{\alpha(X^4 + Y^4)^2(21U^2 + V^2) = T^2(C^2 + D^2)(Z^2 - W^2)}\, P$

Explanation: Let x = $k^n, y = k^{n+1}, z = k^{n+3}, w = k^{n+2}, p = k^{6n}, U = 2^n, V = 2^{n+1}, T = 5(3)^n$

Consider $\alpha(X^4 + Y^4)(\boldsymbol{21U^2 + V^2}) = \alpha k^{8n}(1 + k^4)^2(5(3)^n)^2$

Again consider $(Z^2 - W^2)P = k^{8n}(k^6 - k^4)$.

It follows that $\boldsymbol{\alpha(X^4 + Y^4)^2(21U^2 + V^2) = T^2(C^2 + D^2)(Z^2 - W^2)}\, P$ implies that

$\alpha k^{8n}(1 + k^4)^2(5(3)^n)^2 = (C^2 + D^2)k^{8n}(k^6 - k^4)(5(3)^n)^2$ implies $\alpha(1 + k^4)^2 = (C^2 + D^2)(k^6 - k^4)$.

Solve for α, whenever $(C, D, 1 + k^4)$ is a Pythagorean Triplet.

From the References [1], we know that $(C, D, 1 + k^4)$ becomes a Pythagorean Triplet with $C = (2k^2)$, $D = (k^4 - 1)$, $\boldsymbol{C^2 + D^2 = (1 + k^4)^2}$. Hence $\alpha = (k^6 - k^4)$.

Hence $\alpha(\boldsymbol{X^4 + Y^4})^2(\boldsymbol{21U^2 + V^2}) = \boldsymbol{T^2}(C^2 + D^2)(Z^2 + W^2)P$ having integer design of solution is parameterized by integers k and n, with variables defined as:

x = $k^n, y = k^{n+1}, z = k^{n+3}, w = k^{n+2}, p = k^{6n}, \alpha = (k^6 - k^4), C = 2k^2, D = k^4 - 1$,

$U = 2^{n+1}$, $V = 2^n$ and $T = 5(3)^n$

Verification: Consider **LHS**

$\alpha(\boldsymbol{X^4 + Y^4})^2(\boldsymbol{6U^2 + V^2}) = (k^6 - k^4)(\boldsymbol{k^{4n} + k^{4n+4}})^2(5(3)^n)^2 = \boldsymbol{k^{8n}}(k^6 - k^4)(\boldsymbol{1 + k^4})^2(5(3)^n)^2$.

Consider RHS

$\boldsymbol{T^2}(C^2 + D^2)(Z^2 - W^2)P = (5(3)^n)^2(\boldsymbol{1 + k^4})^2(k^{2n+6} - k^{2n+4})k^{6n} = \boldsymbol{k^{8n}}(k^6 - k^4)(\boldsymbol{1 + k^4})^2(5(3)^n)^2$.

Hence LHS $= \boldsymbol{RHS}$.

Proportion 2: A Study on exponential integer solution of above Diophantine Equation at

$\boldsymbol{\beta = 2}$ **is** $\alpha(\boldsymbol{X^4 + Y^4})^2(\boldsymbol{21U^2 + V^2}) = \boldsymbol{T^2}(C^2 + D^2)(Z^2 - W^2)P^2$

Explanation: Let x = $k^n, y = k^{n+1}, z = k^{n+3}, w = k^{n+2}, p = k^{3n}, U = 2^n, V = 2^{n+1}, T = 5(3)^n$

Consider $\alpha(\boldsymbol{X^4 + Y^4})^2(\boldsymbol{21U^2 + V^2}) = \alpha k^{8n}(1 + k^4)^2(5(3)^n)^2$.

Again consider $(Z^2 - W^2)\boldsymbol{P^2} = k^{8n}(k^6 - k^4)$.

It follows that $\alpha(\boldsymbol{X^4 + Y^4})^2(\boldsymbol{21U^2 + V^2}) = \boldsymbol{T^2}(C^2 + D^2)(Z^2 - W^2)P^2$ implies that

$\alpha k^{8n}(1 + k^4)^2(5(3)^n)^2 = (C^2 + D^2)k^{8n}(k^6 - k^4)(5(3)^n)^2$ implies $\alpha(1 + k^4)^2 = (C^2 + D^2)(k^6 - k^4)$.
Solve for α, whenever $(C, D, 1 + k^4)$ is a Pythagorean Triplet.

From the References [2], we know that $(C, D, 1 + k^4)$ becomes a Pythagorean Triplet

with $C = (2k^2)$, $D = (k^4 - 1)$, $\boldsymbol{C^2 + D^2 = (1 + k^4)^2}$Hence $\alpha = (k^6 - k^4)$.

Verification: Consider **LHS**

$\alpha(\boldsymbol{X^4}+\boldsymbol{Y^4})^2(\boldsymbol{21U^2}+\boldsymbol{V^2}) = (k^6+k^4)(\boldsymbol{k^{4n}}+\boldsymbol{k^{4n+4}})^2(5(3)^n)^2 = (5(3)^n)^2\boldsymbol{k^{8n}}(k^6-k^4)(\boldsymbol{1}+\boldsymbol{k^4})^2.$

Consider RHS

$\boldsymbol{T^2}(C^2+D^2)(Z^2-W^2)P^2 = (5(3)^n)^2(\boldsymbol{1}+\boldsymbol{k^4})^2(k^{2n+6}-k^{2n+4})k^{6n} = (5(3)^n)^2\boldsymbol{k^{8n}}(k^6-k^4)(\boldsymbol{1}+\boldsymbol{k^4})^2.$

Hence LHS $= \boldsymbol{RHS}.$

Proportion 3: A Study on exponential integer solution of above Diophantine Equation at

$\boldsymbol{\beta = 3}$ **is** $\boldsymbol{\alpha(X^4+Y^4)^2(21U^2+V^2) = T^2(C^2+D^2)(Z^2-W^2)P^3}$

Explanation: Let x = $k^n, y = k^{n+1}, z = k^{n+3}, w = k^{n+2}, p = k^{2n}, U = 2^n, V = 2^{n+1}, T = 5(3)^n$.

Consider $\alpha(\boldsymbol{X^4}+\boldsymbol{Y^4})^2(\boldsymbol{21U^2}+\boldsymbol{V^2}) = \alpha k^{8n}(1+k^4)^2(5(3)^n)^2$.

Again consider $(Z^2-W^2)\boldsymbol{P^3} = k^{8n}(k^6-k^4)$.

It follows that $\alpha(\boldsymbol{X^4}+\boldsymbol{Y^4})^2(\boldsymbol{21U^2}+\boldsymbol{V^2}) = \boldsymbol{T^2}(\boldsymbol{C^2}+\boldsymbol{D^2})(\boldsymbol{Z^2}-\boldsymbol{W^2})\boldsymbol{P^3}$ implies that

$\alpha k^{8n}(1+k^4)^2(5(3)^n)^2 = (5(3)^n)^2(C^2+D^2)k^{8n}(k^6-k^4)$ implies $\alpha(1+k^4)^2 = (C^2+D^2)(k^6-k^4)$.
Solve for α, whenever $(C, D, 1+k^4)$ is a Pythagorean Triplet.

From the References [3], we know that $(C, D, 1+k^4)$ becomes a Pythagorean Triplet

with $C = (2k^2)$, $D = (k^4-1)$, $\boldsymbol{C^2}+\boldsymbol{D^2} = (\boldsymbol{1}+\boldsymbol{k^4})^2$Hence $\alpha = (k^6-k^4)$.

Verification: Consider **LHS**

$\alpha(\boldsymbol{X^4}+\boldsymbol{Y^4})^2(\boldsymbol{U^2}+\boldsymbol{V^2}) = (k^6-k^4)(\boldsymbol{k^{4n}}+\boldsymbol{k^{4n+4}})^2(5(3)^n)^2 = \boldsymbol{k^{8n}}(k^6-k^4)(\boldsymbol{1}+\boldsymbol{k^4})^2\,(5(3)^n)^2.$

Consider RHS

$\boldsymbol{T^2}(C^2+D^2)(Z^2-W^2)P^3 = (5(3)^n)^2\,(\boldsymbol{1}+\boldsymbol{k^4})^2(k^{2n+6}-k^{2n+4})k^{6n} = (5(3)^n)^2\boldsymbol{k^{8n}}(k^6-k^4)(\boldsymbol{1}+\boldsymbol{k^4})^2.$

Hence LHS $= \boldsymbol{RHS}.$

Proportion 4 A Study on exponential integer solution of above Diophantine Equation at

$\boldsymbol{\beta = 4}$ **is** $\boldsymbol{\alpha(X^4+Y^4)^2(21U^2+V^2) = T^2(C^2+D^2)(Z^2-W^2)P^4.}$

Explanation: Let x = $k^n, y = k^{n+1}, z = k^{2n+3}, w = k^{2n+2}, p = k^n, U = 2^n, V = 2^{n+1}, T = 5(3)^n$

Consider $\alpha(\boldsymbol{X^4}+\boldsymbol{Y^4})^2(\boldsymbol{21U^2}+\boldsymbol{V^2}) = \alpha k^{8n}(1+k^4)^2\,(5(3)^n)^2$.

Again consider $(Z^2-W^2)\boldsymbol{P^4} = k^{8n}(k^6-k^4)$.

It follows that $\boldsymbol{\alpha(X^4+Y^4)^2(21U^2+V^2) = T^2(C^2+D^2)(Z^2-W^2)P^4}$ implies that

$\alpha k^{8n}(1+k^4)^2(5(3)^n)^2 = (5(3)^n)^2(C^2+D^2)k^{8n}(k^6-k^4)$ implies $\alpha(1+k^4)^2 = (C^2+D^2)(k^6-k^4)$.
Solve for α, whenever $(C, D, 1+k^4)$ is a Pythagorean Triplet.

From the References [4],[5], we know that

$(C, D, 1+k^4)$ becomes a Pythagorean Triplet with $C = (2k^2)$, $D = (k^4-1)$,

$\boldsymbol{C^2}+\boldsymbol{D^2} = (\boldsymbol{1}+\boldsymbol{k^4})^2$Hence $\alpha = (k^6-k^4)$.

Verification: Consider **LHS**

is $\boldsymbol{\alpha(X^4+Y^4)^2(21U^2+V^2)} = (k^6-k^4)(\boldsymbol{k^{4n}}+\boldsymbol{k^{4n+4}})^2(5(3)^n)^2 = \boldsymbol{k^{8n}}(k^6-k^4)(\boldsymbol{1}+\boldsymbol{k^4})^2\,(5(3)^n)^2$

Consider RHS

$$\boldsymbol{T^2(C^2+D^2)(Z^2-W^2)P^4} = (5(3)^n)^2\ (\boldsymbol{1+k^4})^2(k^{2n+6}-k^{2n+4})k^{6n} = \boldsymbol{k^{8n}}(k^6-k^4)(\boldsymbol{1+k^4})^2(5(3)^n)^2$$

Hence LHS $= \boldsymbol{RHS}$.

Proportion 5: A Study on exponential integer solution of above Diophantine Equation at

$\boldsymbol{\beta = 5}$ **is** $\boldsymbol{\alpha(X^4+Y^4)^2(21U^2+V^2) = T^2(C^2+D^2)(Z^2-W^2)P^5}$.

Explanation: Let x $= k^n, y = k^{n+1}, z = k^{2n+3}, w = k^{2n+2}, p = k^n, U = 2^n, V = 2^{n+1},\ T = 5(3)^n$

Consider $\alpha(\boldsymbol{X^4+Y^4})^2(\boldsymbol{6U^2+V^2}) = \alpha k^{8n}(1+k^4)^2\ (5(3)^n)^2$.

Again consider $(Z^2-W^2)\boldsymbol{P^5} = k^{9n}(k^6-k^4)$.

It follows that $\boldsymbol{\alpha(X^4+Y^4)^2(21U^2+V^2) = T^2(C^2+D^2)(Z^2-W^2)P^5}$ implies that

$\alpha k^{8n}(1+k^4)^2(5(3)^n)^2 = (5(3)^n)^2(C^2+D^2)k^{9n}(k^6-k^4)$ implies $\alpha(1+k^4)^2 = (C^2+D^2)k^n(k^6-k^4)$. Solve for α, whenever $(\ C, D, 1+k^4)$ is a Pythagorean Triplet.

From the References [6],[7],[8], we know that

$(\ C, D, 1+k^4)$ becomes a Pythagorean Triplet with $C = (2k^2\ ),\ D = (k^4-1)$,

$\boldsymbol{C^2+D^2} = (\boldsymbol{1+k^4})^2$Hence $\alpha = k^n(k^6-k^4)$.

Verification: Consider **LHS**

$$\alpha(\boldsymbol{X^4+Y^4})^2(\boldsymbol{21U^2+V^2}) = k^n\ (k^6-k^4)(\boldsymbol{k^{4n}+k^{4n+4}})^2(5(3)^n)^2 = (5(3)^n)^2\boldsymbol{k^{9n}}(k^6-k^4)(\boldsymbol{1+k^4})^2.$$

Consider RHS

$$\boldsymbol{T^2}(C^2+D^2)(Z^2-W^2)P^5 = (5(3)^n)^2(\boldsymbol{1+k^4})^2(k^{4n+6}-k^{4n+4})k^{5n} = (5(3)^n)^2\boldsymbol{k^{9n}}(k^6-k^4)(\boldsymbol{1+k^4})^2.$$

Hence LHS $= \boldsymbol{RHS}$.

Proportion 6: A Study on exponential integer solution of above Diophantine Equation at

$\boldsymbol{\beta = 6}$ **is** $\boldsymbol{\alpha(X^4+Y^4)^2(21U^2+V^2) = T^2(C^2+D^2)(Z^2-W^2)P^6}$.

Explanation: Let x $= k^n, y = k^{n+1}, z = k^{2n+3}, w = k^{2n+2}, p = k^n, U = 2^n, V = 2^{n+1}$,

$T = 5(3)^n$Consider $\alpha(\boldsymbol{X^4+Y^4})^2(\boldsymbol{6U^2+V^2}) = \alpha k^{8n}(1+k^4)^2(5(3)^n)^2$.

Again consider $(Z^2-W^2)\boldsymbol{P^6} = k^{10n}(k^6-k^4)$.

It follows that $\boldsymbol{\alpha(X^4+Y^4)^2(21U^2+V^2) = T^2(C^2+D^2)(Z^2-W^2)P^6}$ implies that

$\alpha k^{8n}(1+k^4)^2(5(3)^n)^2 = (5(3)^n)^2(C^2+D^2)k^{10n}(k^6-k^4)$ implies

$\alpha(1+k^4)^2 = (C^2+D^2)k^{2n}(k^6-k^4)$. Solve for α, whenever $(\ C, D, 1+k^4)$ is a Pythagorean Triplet.

From the References [8], we know that

$(\ C, D, 1+k^4)$ becomes a Pythagorean Triplet with $C = (2k^2\ ),\ D = (k^4-1)$,

$\boldsymbol{C^2+D^2} = (\boldsymbol{1+k^4})^2$Hence $\alpha = k^{2n}(k^6-k^4)$.

Verification: Consider **LHS**

$$\alpha(\boldsymbol{X^4+Y^4})^2(\boldsymbol{21U^2+V^2}) = k^{2n}\ (k^6-k^4)(\boldsymbol{k^{4n}+k^{4n+4}})^2(5(3)^n)^2 = (5(3)^n)^2\boldsymbol{k^{10n}}(k^6-k^4)(\boldsymbol{1+k^4})^2.$$

Consider RHS

$$\boldsymbol{T^2(C^2+D^2)(Z^2-W^2)P^6} = (5(3)^n)^2(\boldsymbol{1+k^4})^2(k^{4n+6}-k^{4n+4})k^{6n} = (5(3)^n)^2\boldsymbol{k^{10n}}(k^6-k^4)(\boldsymbol{1+k^4})^2.$$

Hence LHS $= \boldsymbol{RHS}$.

Proportion 7: A Study on exponential integer solution of above Diophantine Equation at

$\boldsymbol{\beta = 7}$ **is** $\boldsymbol{\alpha(X^4+Y^4)^2(21U^2+V^2) = T^2(C^2+D^2)(Z^2-W^2)P^7.}$

Explanation: Let x = $k^n, y = k^{n+1}, z = k^{2n+3}, w = k^{2n+2}, p = k^n, U = 2^n, V = 2^{n+1}, T = 5(3)^n$

Consider $\alpha(\boldsymbol{X^4+Y^4})^2(\boldsymbol{21U^2+V^2}) = \alpha k^{8n}(1+k^4)^2(5(3)^n)^2$.

Again consider $(Z^2-W^2)\boldsymbol{P^6} = k^{10n}(k^6-k^4)$.

It follows that $\boldsymbol{\alpha(X^4+Y^4)^2(21U^2+V^2) = T^2(C^2+D^2)(Z^2-W^2)P^7.}$ implies that

$\alpha k^{8n}(1+k^4)^2(5(3)^n)^2 = (5(3)^n)^2(C^2+D^2)k^{11n}(k^6-k^4)$ implies

$\alpha(1+k^4)^2 = (C^2+D^2)k^{3n}(k^6-k^4)$. Solve for α, whenever $(C, D, 1+k^4)$ is a Pythagorean Triplet.

From the References [7],[8], we know that

$(C, D, 1+k^4)$ becomes a Pythagorean Triplet with $C = (2k^2)$, $D = (k^4-1)$,

$\boldsymbol{C^2+D^2 = (1+k^4)^2.}$ Hence $\alpha = k^{3n}(k^6-k^4)$.

Verification: Consider **LHS**

$\boldsymbol{\alpha(X^4+Y^4)^2(21U^2+V^2)} = k^{3n}\,(k^6-k^4)(\boldsymbol{k^{4n}+k^{4n+4}})^2(5(3)^n)^2 = (5(3)^n)^2\,\boldsymbol{k^{11n}}(k^6-k^4)(\boldsymbol{1+k^4})^2.$

Consider RHS

$\boldsymbol{T^2(C^2+D^2)(Z^2-W^2)P^7} = (5(3)^n)^2(\boldsymbol{1+k^4})^2(k^{4n+6}-k^{4n+4})k^{7n} = (5(3)^n)^2\,\boldsymbol{k^{11n}}(k^6-k^4)(\boldsymbol{1+k^4})^2.$

Hence LHS $= \boldsymbol{RHS.}$

IV. Main Result:

A Study on exponential integer solution of above Diophantine Equation at

$\boldsymbol{\alpha(X^4+Y^4)^2(21U^2+V^2) = T^2(C^2+D^2)(Z^2-W^2)P^\beta.}$

Explanation: Let x = $k^n, y = k^{n+1}, z = k^{2n+3}, w = k^{2n+2}, p = k^n$,

$U = 2^{n+1},\ V = 2^n, T = 5(2)^n$Consider $\alpha(\boldsymbol{X^4+Y^4})^2(\boldsymbol{6U^2+V^2}) = \alpha k^{8n}(1+k^4)^2\,(5(3)^n)^2$.

Again consider $(Z^2-W^2)\boldsymbol{P^\beta} = k^{4n+n\beta}(k^6-k^4)$.

It follows that $\boldsymbol{\alpha(X^4+Y^4)^2(21U^2+V^2) = T^2(C^2+D^2)(Z^2-W^2)P^\beta.}$

implies that

$\alpha k^{8n}(1+k^4)^2(5(3)^n)^2 = (5(3)^n)^2(C^2+D^2)k^{4n+n\beta}(k^6-k^4)$ implies

$\alpha(1+k^4)^2 = (C^2+D^2)k^{-4n+n\beta}(k^6+k^4)$. Solve for α, whenever $(C, D, 1+k^4)$ is a Pythagorean Triplet.

From the References [8], we know that

$(C, D, 1+k^4)$ becomes a Pythagorean Triplet with $C = (2k^2)$, $D = (k^4-1)$,

$\boldsymbol{C^2+D^2 = (1+k^4)^2.}$ Hence $\alpha = k^{-4n+n\beta}(k^6-k^4) = k^{(\beta-4)n}(k^6-k^4)$.

Verification: Consider **LHS**

$\alpha(\boldsymbol{X^4+Y^4})^2(\boldsymbol{21U^2+V^2}) = k^{(\beta-4)n}(k^6-k^4)\,(\boldsymbol{k^{4n}+k^{4n+4}})^2(5(3)^n)^2$

$= (5(3)^n)^2\boldsymbol{k^{(\beta+4)n}}(k^6-k^4)(\boldsymbol{1+k^4})^2.$

Consider RHS

$$\boldsymbol{T^2(C^2+D^2)(Z^2-W^2)P^\beta} = (5(3)^n)^2(\mathbf{1}+\boldsymbol{k^4})^2(k^{4n+6}-k^{4n+4})k^{\beta n}$$

$$= (5(3)^n)^2\boldsymbol{k}^{(\beta+4)n}(k^6-k^4)(\mathbf{1}+\boldsymbol{k^4})^{\mathbf{2}}.$$

Hence LHS $= \boldsymbol{RHS}$.

V. Conclusion:

This equation generalizes classical Diophantine problems, blending sums of fourth powers with multiplicative factorizations. While challenging, targeted parametrization and modular analysis can yield solutions. Future work may classify solutions for specific α, β or link to broader number-theoretic frameworks. The parametric framework provides infinite families of solutions by exploiting algebraic identities and modular arithmetic. Future work could explore non-parametric solutions or generalizations to higher exponents.

Integer design of solutions in complex stoichiometric reaction systems involves systematically formulating and solving the atom-balance constraints as systems of integer linear equations, followed by computational approaches to identify minimal, feasible integer stoichiometric coefficients that satisfy conservation laws and additional constraints

This synthesis of algebraic number theory and chemical analysis provides robust, computationally efficient, and theoretically complete modeling tools. Integer solutions deliver not just chemical consistency but also reliability in detecting chemically feasible reaction mechanisms and molecular structures.

In summary, systems of Diophantine equations are both a theoretical and practical powerhouse for balancing complex chemical equations and determining molecular formulas when integer consistency is essential. Methods such as induction, trial computation, and number-theory-based parametrization—often leveraging patterns like Pythagorean triples—allow chemists and mathematicians to ensure atomic conservation and design integer-based models with a high degree of rigor and efficiency

This paper focused on a study to find integer design of solutions Diophantine Equation

$\boldsymbol{\alpha(X^4+Y^4)^2(21U^2+V^2)=T^2(C^2+D^2)(Z^2-W^2)P^\beta}$ **With** $\boldsymbol{\alpha>0, \gamma=2,3, \beta=1,2,3,4,5,6,7\ and}$

$\boldsymbol{x<y<w<z}$ with Mathematical induction & generation of Pythagorean triplets.

for $\boldsymbol{\beta=1}$, having integer design of solution is parameterized by positive integers k and n, with variables defined as:

x = $k^n, y=k^{n+1}, z=k^{n+3}, w=k^{n+2}, U=2^n, V=2^{n+1}, T=5(3)^n,\ p=k^{6n}, \alpha=(k^6-k^4), C=2k^2,$

$D=k^4-1.$

for $\boldsymbol{\beta=2}$, having integer design of solution is parameterized by integers k and n, with variables defined as:

x = $k^n, y=k^{n+1}, z=k^{n+3}, w=k^{n+2}, U=2^n, V=2^{n+1}, T=5(3)^n, p=k^{3n}, \alpha=(k^6-k^4), C=2k^2,$

$D=k^4-1.$

for $\boldsymbol{\beta=3}$, having integer design of solution is parameterized by integers k and n, with variables defined as:

x = $k^n, y=k^{n+1}, z=k^{n+3}, w=k^{n+2}, U=2^n, V=2^{n+1}, T=5(3)^n, p=k^{2n}, \alpha=(k^6-k^4), C=2k^2,$

$D=k^4-1.$

for $\boldsymbol{\beta=4}$, having integer design of solution is parameterized by integers k and n, with variables defined as:

x = $k^n, y = k^{n+1}, z = k^{2n+3}, w = k^{2n+2}, U = 2^n, V = 2^{n+1}, T = 5(3)^n, p = k^n, \alpha = (k^6 - k^4), C = 2k^2$, $D = k^4 - 1$.

for $\beta = 5$, having integer design of solution is parameterized by integers k and n, with variables defined as:

x = $k^n, y = k^{n+1}, z = k^{n+3}, w = k^{n+2}, U = 2^n, V = 2^{n+1}, T = 5(3)^n, p = k^{2n}, \alpha = k^n(k^6 - k^4)$, $C = 2k^2, D = k^4 - 1$.

for $\beta = 6$, having integer design of solution is parameterized by integers k and n, with variables defined as:

x = $k^n, y = k^{n+1}, z = k^{2n+3}, w = k^{2n+2}, U = 2^n, V = 2^{n+1}, T = 5(3)^n, p = k^n, \alpha = k^{2n}(k^6 - k^4), C = 2k^2, D = k^4 - 1$.

for $\beta = 7$, having integer design of solution is parameterized by integers k and n, with variables defined as:

x = $k^n, y = k^{n+1}, z = k^{2n+3}, w = k^{2n+2}, U = 2^n, V = 2^{n+1}, T = 5(3)^n, p = k^n, \alpha = k^{3n}(k^6 - k^4)$, $C = 2k^2, D = k^4 - 1$.

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