Existence and uniqueness of solutions for fuzzy differential equations via common fixed point theorems in G-fuzzy metric space .

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Abstract:- This work investigates the existence and uniqueness of solutions for fuzzy differential equations using common fixed point theorems in G-fuzzy metric spaces. By converting such equations into integral operator forms , we estabalish generalized contractive conditions ensuring unique fuzzy solutions. Illustrative examples demonstrate the practical relevance of the results in modelling uncertain systems .

Keywords:- G-fuzzy metric space , common fixed point , fuzzy differential equations , existence and uniqueness .

1. **Introduction**:- fuzzy sets , introduced by Zadeh [13] , enable the modelling of uncertainty in real -world problems . Extending classical concepts , fuzzy metric spaces [3,4] have become essential for studying systems where data is imprecise . Fuzzy differential equations model such dynamic systems but often face challenges regarding existence and uniqueness of solutions [6]. Fixed point theory , especially in fuzzy metric spaces [5], offers powerful tools to resolve these challenges . This paper presents new existence and uniqueness results for fuzzy differential equations via common fixed point theorems in G-fuzzy metric spaces , contributing to both theory and practical applications .

2. **Definitions**:-

2.1 Fuzzy set [11,13] :- A fuzzy set A in a set X is a mapping $\mu_A : X \to [0,1]$, where $\mu_A(x)$ denotes the degree of membership of x in A.

2.2 Fuzzy number:- A fuzzy number ã is a fuzzy set on R which is normal , convex , upper semi-continuous, and has bounded support .

2.3 t-norm :- A t-norm is a function T: $[0,1] \times [0,1] \to [0,1]$ satisfying:

- Commutativity : T(a ,b) = T(b ,a)
- Associativity : T(a, T(b ,c)) = T (T(a ,b) ,c)
- Monotonicity : if $a \leq c$ and $b \leq d$, then T (a ,b) ≤ T(c ,d)
- Identity : T(a,1) = a

Example : T(a ,b) = a x b

2.4 G – fuzzy metric space [3,8] :- A triple (X,G,T) is called a G–fuzzy metric space if :-

- X is a non empty set .
- G : X x X x (0,∞) →[0,1] satisfies for all x ,y ,z ∈ X and s ,t >0
  a. G(x ,y ,t) = 1 if and only if x=y .
  b. G(x ,y ,t) =G(y ,x ,t) .
  c. G(x ,z ,t +s) ≥ T(G(x ,y ,t), G(y ,z ,s)) .

2.5 Cauchy sequence in G – fuzzy metric space :- A sequence $\{x_n\}$ in X is G – Cauchy if for every ε>0 and t>0 , there exists $N \in \mathbb{N}$ such that for all m,n > N :-

$G(x_n , x_m ,t) > 1- \varepsilon$.

2.6 Complete G-Fuzzy metric space :- A G -fuzzy metric space (X,G,T) is complete if every G-cauchy sequence converges to a point in X.

2.7 Fixed point :- An element x ε X is a fixed point of a mapping f : X →X if :-

f(x) = x.

2.8 Common fixed point :- An element x ∈ X is a common fixed point of mappings f , g : X→X if :-

f(x) = g(x) = x .

2.9 Fuzzy differential equation [7,12] :- A fuzzy differential equation is of the form $\frac{dy}{dt}$ = f(t,y(t)) , Where y(t) is a fuzzy – valued function.

2.10 Contraction mapping in G- fuzzy metric space [9] :- A mapping f : x→ x is a fuzzy contraction if there exists k ∈ (0,1) such that :-

$G (f(x) , f(y) , t) \geq \psi(G(x ,y ,t) ,$

Where ψ is a function satisfying certain contractive conditions .

Example :- **(linear contraction in G -fuzzy metric space)**

Let X = R and define the G – fuzzy metric as

$G (x ,y ,t) = \frac{t}{t+|x-y|} \quad \forall x ,y \in R , t>0 .$

Let the mapping f: x →x be defined by $f(x) = \frac{3}{4}x$.

Then for any x ,y ∈R ,we have

$| f(x) - f(y) | = | \frac{3}{4}x - \frac{3}{4}y | = \frac{3}{4} | x-y | .$

Since $\frac{3}{4}<1$, the mappings f is a contraction .

Further , in the G -fuzzy metric space , we have

$G (f(x) , f(y) , t) = \frac{t}{t+3/4|x-y|}$

Clearly, $G (f(x) , f(y) , t) > \frac{t}{t+|x-y|} \times \frac{2}{3}$,

Showing that f satisfies a generalized contractive condition in (X , G , T)

Consider the fuzzy differential equation $\frac{dy}{dt} = -\frac{3}{4}\, y(t)$, $y(0) = \tilde{a}$, Where ã is a fuzzy number

Its solution is $y(t) = \tilde{a} \times e^{-\frac{3}{4}t}$.

Equivalently , it can be written in operator form as

$(Fy)(t) = \tilde{a} + \int_0^t -\frac{3}{4} y(s)\, ds$.

Since f is a contraction in the G – fuzzy metric space , the operator F has a unique fixed point corresponding to a unique fuzzy solution of the differential equation . therefore, the existence and uniqueness of fuzzy solution are ensured via the common fixed point theorem in the G- fuzzy metric space .

Remark :- Every contraction mapping defined in a G- fuzzy metric space ensures the existence of a unique fixed point under suitable conditions . This property is crucial in establishing the existence and uniqueness of solutions for fuzzy differential equations , since the solution can often be formulated as the fixed point of an associated integral operator in the G -fuzzy metric space .

2.11 Operator form of fuzzy differential equation [1,10] :- A fuzzy differential equation of the form

$$\frac{dy}{dt} = f\left(t, y(t)\right),$$

With fuzzy initial condition $y(t_0) = \tilde{a}$, can be reformulated as an operator equation :-

$$(Fy)(t) = \tilde{a} + \int_{t_0}^{t} f\left(s, y(s)\right) ds .$$

Here , F is an operator that maps fuzzy-valued functions into fuzzy-valued functions . The fixed points of F are the fuzzy solutions of the differential equation.

3. **Main theorem** :- Let (X,G,T) be a complete G- fuzzy metric space . Let $f:x \to x$ be a mapping satisfying the following condition :-

$G(f(x), f(y), t) \geq \psi(G(x, y, t))$ for all $x, y \in X$ and $t>0$,where $\psi : [0,1] \to [0,1]$ is a continuous , non decreasing function such that $\psi(s)>s$ for all $s \in [0,1)$.

Then F has a unique fixed point in X .

Moreover , if F is the operator defined by

$$(Fy)(t) = y_0 + \int_{t_0}^{t} f\left(y(s)\right) ds ,$$

Then the fuzzy differential equation

$$\frac{dy}{dt} = f(y(t)) , \quad y(t_0) = y_0$$

Has a unique fuzzy solution in X .

Proof :- let $x0 \in X$ be arbitrary . define a sequence x_n in X by

$$x_{n+1} = f(x_n) , \quad n = 0,1,2, \ldots\ldots\ldots$$

Since f satisfies $G(f(x), f(y), t) \geq \psi(G(x, y, t))$ for all $x, y \in X$ and $t>0$, we have

$$G(x_{n+1}, x_n, t) = G(f(x_n), f(x_{n-1}), t) \geq \psi(G(x_n, x_{n-1}, t)) .$$

Because $\psi(s)>s$ for all $s \in [0,1)$, it follows that the sequence $G(x_n, x_{n-1}, t)$ is increasing and bounded above by 1 . hence ,

$$\lim_{n\to\infty} G(x_{n+1}, x_n, t) = 1 .$$

We now prove that x_n is a G – Cauchy sequence .

For m>n , by the properties of G and t -norm T , we have

$$G(x_n, x_m, t) \geq T(G(x_n, x_{n+1}, t), G(x_{n+1}, x_{n+2}, t), \ldots\ldots\ldots\ldots, G(x_{m-1}, x_m, t)) .$$

Since each term G(x_k , x_{k+1} , t) tends to 1 as k $\rightarrow\infty$, their t-norm also tends to 1 . thus , for any ε>0 , there exists N ∈ ℕ such that for all m,n> N ,

G(x_n , x_m , t) > 1-ε . Therefore , x_n is G – cauchy .

Since (X , G ,T) is complete , there exists x* ∈ X such that $\lim_{n\to\infty} x_n = x^*$.

Next we show that x* is a fixed point of f .

Observe that G(f(x*) , x*, t) = $\lim_{n\to\infty} G(f(x^*), f(x_n)\,, t) \geq \lim_{n\to\infty} \psi(G(x^*, x_{n\,,}\, t))$ = ψ(1) = 1

Hence , G(f(xt)) = 1 . Therefore , f(x*) = x* .

For uniqueness , suppose y* ≠ x* is another fixed point . Then

G(f(x*), f(y*), t) = G(x* , y* , t) ≥ ψ(G(x* , y* , t)) .

But since ψ(s)>s for all s ∈ [0,1) ,

This inequality holds only if G(xt) = 1 .

Thus , x* = y* . hence the fixed point is unique .

Now consider the operator

(Fy)(t) = $y_0 + \int_{t_0}^{t} f(y(s))ds$.

This operator maps fuzzy valued function into themselves. Under the contraction condition on f , it can be shown similarly that F is a contraction mapping in the space of fuzzy – valued continuous functions equipped with a suitable G – fuzzy metric . Therefore , F has a unique fixed point , which corresponds to the unique fuzzy solution of the differential equation

$\frac{dy}{dt}$ = f(y(t)) , y(t_0) = y_0 .

This completes the proof .

Example :- consider X = ℝ with G -fuzzy metric G(x ,y ,t) = $\frac{t}{t+|x-y|}$, t>0 .

Define f(x) = $\frac{3}{4}x$. then , for any x ,y ∈ ℝ ,

| f(x) -f(y) | = $\frac{3}{4}|x-y|$.

Hence , G(f(x), f(y) , t) = $\frac{t}{t+3/4|x-y|}$ >$\frac{3}{4}$. G(x ,y ,t) .

Define ψ(s) = $\frac{3}{4}$s . since ψ(s)>s for all s ∈[0,1) , satisfies the contractive condition of theorem 3 .

Thus , f has the unique fixed point x = 0 .

The fuzzy differential equation $\frac{dy}{dt} = -\frac{3}{4}y(t)$ has the unique fuzzy solution y(t) = $\tilde{a}e^{-\frac{3}{4}t}$

Corollary :- let (ℝ,G,T) be the G-fuzzy metric space where G(x ,y ,t) = $\frac{t}{t+|x-y|}$, t>0 ,

And let f(x) = kx for some constant k with |k| <1 . then f has a unique fixed point at x=0 .

Moreover the fuzzy differential equation $\frac{dy}{dt}$ = ky(t) , y(0) = ã , has the unique fuzzy solution y(t) = $\tilde{a}e^{kt}$.

4. **Application** [2]:- In many real -life situations like population growth , economic planning , or disease spread , exact values of rates and parameters are often uncertain . G-fuzzy metric spaces and fixed point results help ensure that , even under such fuzziness , models give unique and stable solutions . This makes predictions more reliable and supports better decision – making despite uncertain data .
5. **Conclusion**:- In this paper , we established new existence and uniqueness results for fuzzy differential equations using common fixed point theorems in G-fuzzy metric spaces . The presented theorems extend known results and provide a powerful framework to handle uncertainty in various practical problems. Our examples and applications demonstrate how these theoretical results can ensure stable and unique solutions in systems affected by imprecise data , such as population dynamics or economic forecasting . Future research may explore more generalized conditions or apply these results to complex systems in science and engineering .

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