Coupled Fixed Point Theorems for Mixed Monotone Mappings with Applications

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Abstract:

We obtain coupled fixed-point theorems for mixed monotone mapping and finally established some applications these are the generalizations of the results of the Hemant Kumar Nashine and Wasfi Shatanawi [4].

Key words: Complete metric space, monotone mappings, altering distances, coupled coincidence point, Partial ordered set.

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1. Introduction:

Fixed point theory shows a important role in many branches of mathematical analysis and its applications, mainly in the study of nonlinear analysis, differential equations, and dynamic systems. In recent years, significant attention has been devoted to the development of fixed point results in partially ordered metric spaces, especially for mappings that exhibit monotonicity properties.

A notable contribution in this direction was made by Hemant Kumar Nashine and Wasfi Shatanawi [4], who established several fixed point theorems for mixed monotone mappings using the concept of altering distance functions. Their results extended and unified many earlier fixed point theorems, providing a powerful framework for analyzing the existence of fixed and coupled fixed points under weaker contractive conditions.

Motivated by their work, we aim to further generalize these results by developing new coupled fixed point theorems for mixed monotone mappings in partially ordered metric spaces. Our approach is based on refined contractive conditions involving altering distance functions and leverages the mixed monotonicity of the mappings. We also provide illustrative examples and applications to demonstrate the validity and utility of the main results. These generalizations not only encompass the findings of Nashine and Shatanawi but also contribute to the ongoing development of fixed point theory in more generalized settings.

Recently established Hemant Kumar Nashine and Wasfi Shatanawi some fixed point theorems for mixed monotone mappings by using altering distances function [4].

Hemant Kumar Nashine and Wasfi Shatanawi [4] established the following

Theorem 1.1.[4] Let $(X, d, \leq)$ be an ordered metric space. Let $F: X \times X \to X$ and $g: X \to X$ be mappings such that F has the mixed g-monotone property on X such that there exist two elements $x_0, y_0 \in X$ with $g(x_0) \leq F(x_0, y_0)$ and $g(y_0) \geq F(y_0, x_0)$ .Suppose there exist non-negative real numbers α, β, L with $\alpha + \beta < 1$ such that

$$d\big(F(x,y), F(u,v)\big) \leq \big[\alpha min\big\{d\big(F(x,y), g(x)\big), d\big(F(u,v), g(x)\big)\big\} + \beta min\big\{d\big(F(x,y), g(u)\big), d\big(F(u,v), g(u)\big)\big\} + L min\big\{d\big(F(x,y), g(u)\big), d\big(F(u,v), g(x)\big)\big\}\big]$$

for all $(x, y), (u, v) \in X \times X$ with $g(x) \leq g(u)$ and $g(y) \geq g(v)$. Further suppose $F(X \times X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X. Also, suppose that X satisfies the following properties

1. If a non-decreasing sequence $\{x_n\}$ in X converges to $x \in X$ then $x_n \leq x$ for all n,
2. If a non-increasing sequence $\{y_n\}$ in X converges to $y \in X$,then $y_n \geq y$ for all n.
   Then there exist $x, y \in X$ such that $F(x, y) = g(x)$ and $F(y, x) = g(y)$ that is, F and g have a coupled coincidence point $(x, y) \in X \times X$

Similarly we generalize the remaining results of Hemant Kumar Nashine and Wasfi Shatanawi [4]

Delbosco[1] and Skof[3] introduced a new concept known as altering distances

Definition 1.1.([1] and[3])

A function $\psi: R^+ \to R^+$ is said to be an altering distances function if the following properties are satisfied

1. ψ is continuous and strictly increasing on R^+,
2. $\psi(t) = 0$ if and only if t=0 and

3. $\psi(t) \geq Mt^{\mu}$ for every $t > 0$where $\mu > 0$ and $M > 0$ are constants.

Note that such ψ is not necessarily a metric.

For example, $\psi(t) = t^2$

Definition 1.2 ([2]). Let $(X, \leq)$ be a partially ordered set and $F: X \times X \to X$. The mapping F said to have the mixed monotone property if $F(X, Y)$ is monotone non-decreasing in X and monotone non-increasing in y,that is, for any $x, y \in X$,

$$x_1, x_2 \in X, \qquad x_1 \leq x_2 \Longrightarrow F(x_1, y) \leq F(x_2, y)$$

and

$$y_1, y_2 \in X, \qquad y_1 \leq y_2 \Longrightarrow F(x, y_1) \leq F(x, y_2).$$

Definition 1.2 ([2]). An element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping $F: X \times X \to X$ if $F(x, y) = x$ and $F(y, x) = y$.

Theorem 1.3([4]). Let Θ denote the class of those functions $\theta: [0, +\infty)^2 \to [0,1)$ which satisfies the condition for any sequences $\{t_n\}$, $\{s_n\}$ of positive real numbers

$\theta(t_n, s_n) \to 1$ implies $t_n, s_n \to 0$.

Definition 1.4. Let (X, d) be a metric space and $F: X \times X \to X$ and $g: X \to X$ be mappings. We say F and g commute if $F\big(g(x), g(y)\big) = g\big(F(x, y)\big)$ for all $x, y \in X$.

2. Main Results

Theorem 2.1. Let $(X, d, \leq)$ be an ordered metric space. Let $F: X \times X \to X$ and $g: X \to X$ be mappings such that F has the mixed g-monotone property on X such that there exist two elements $x_0, y_0 \in X$ with $g(x_0) \leq F(x_0, y_0)$ and $g(y_0) \geq F(y_0, x_0)$ .Suppose there exist non-negative real numbers α, β, L with $\alpha + \beta < 1$ such that

$$\begin{aligned} &d\big(F(x,y), F(u,v)\big) + d\big(F(y,x)), F(v,u)\big) \leq \\ &\theta\Big(d\big(g(x), g(u)\big), d\big(g(y), g(v)\big)\Big)\Big[\alpha\big(min\{d\big(F(x,y), g(x)\big), d\big(F(u,v), g(x)\big)\} + \\ &min\{d\big(F(y,x), g(v)\big), d\big(F(v,u), g(v)\big)\}\big) + \beta\big(min\{d\big(F(x,y), g(u)\big), d\big(F(u,v), g(u)\big)\} + \\ &min\{d\big(F(y,x), g(v)\big), d\big(F(v,u), g(v)\big)\}\big) + L\Big(min\{d\big(F(x,y), g(u)\big), d\big(F(u,v), g(x)\big)\} + \\ &min\Big\{d\Big(F(y,x), \big(g(v)\big)\Big), d\big(F(v,u), g(y)\big)\Big\}\Big)\Big] \qquad (2.1.1) \end{aligned}$$

for all $(x, y), (u, v) \in X \times X$ with $g(x) \leq g(u)$ and $g(y) \geq g(v)$. Further suppose $F(X \times X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X. Also, suppose that X satisfies the following properties

1. If a non-decreasing sequence $\{x_n\}$ in X converges to $x \in X$ then $x_n \leq x$ for all n,
2. If a non-increasing sequence $\{y_n\}$ in X converges to $y \in X$,then $y_n \geq y$ for all n.
   Then there exist $x, y \in X$ such that $F(x, y) = g(x)$ and $F(y, x) = g(y)$ that is , F and g have a coupled coincidence point $(x, y) \in X \times X$

Proof: Let $x_0, y_0 \in X$ be such that $g(x_0) \leq F(x_0, y_0)$ and $g(y_0) \geq F(y_0, x_0)$

Since $F(X \times X) \subseteq g(X)$

We can choose $x_1, y_1 \in X$ such that $g(x_1) = F(x_0, y_0)$ and $g(y_1) = F(y_0, x_0)$

In the same way we construct

$g(x_2) = F(x_1, y_1)$ and $g(y_2) = F(y_1, x_1)$

Continuing in this way we construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$g(x_{n+1}) = F(x_n, y_n)$ and $g(y_{n+1}) = F(y_n, x_n)$ for all $n \geq 0$
(2.1.2)

Now prove that for all $n \geq 0$

$g(x_n) \leq g(x_{n+1})$ (2.1.3)

and

$g(y_n) \geq g(y_{n+1})$ (2.1.4)

We shall use the mathematical induction

Let $n = 0$ since $g(x_0) \leq F(x_0, y_0)$ and $g(y_0) \geq F(y_0, x_0)$

In view of $g(x_1) = F(x_0, y_0)$ and $g(y_1) = F(y_0, x_0)$

We have $g(x_0) \leq g(x_1)$ and $g(y_0) \geq g(y_1)$

That is (2.1.3) and (2.1.4) hold for $n = 0$

We assume that (2.1.3) and (2.1.4) hold for some $n > 0$

As F has the mixed g-monotone property and $(x_n) \leq g(x_{n+1})$, $g(y_n) \geq g(y_{n+1})$ from (2.1.2)

we get

$g(x_{n+1}) = F(x_n, y_n) \leq F(x_{n+1}, y_n)$ (2.1.5)

and

$F(y_{n+1}, x_n) \leq F(y_n, x_n) = g(y_{n+1})$ (2.1.6)

Also for the same reason we have

$$g(x_{n+1}) = F(x_{n+1}, y_{n+1}) \geq F(x_{n+1}, y_n) \quad (2.1.7)$$

From (2.1.5) and (2.1.7)

$$g(x_{n+1}) \leq g(x_{n+2})$$

and

$$F(y_{n+1}, x_n) \geq F(y_{n+1}, x_{n+1}) = g(y_{n+2}) \quad (2.1.8)$$

From (2.1.6) and (2.1.8)

$$g(y_{n+1}) \geq g(y_{n+2})$$

Thus by the mathematical induction, we conclude that (2.3) and (2.4) hold for all $n \geq 0$

We check easily that

$$g(x_0) \leq g(x_1) \leq g(x_2) \leq \ldots \leq g(x_{n+1}) \ldots$$

and

$$g(y_0) \geq g(y_1) \geq g(y_2) \geq \ldots \geq g(y_{n+1}) \ldots$$

Assume that there is some $r \in \mathbb{N}$ such that

$$d\big(g(x_r), g(x_{r-1})\big) + d\big(g(y_r), g(y_{r-1})\big) = 0$$

That is $g(x_r) = g(x_{r-1})$ and $g(y_r) = g(y_{r-1})$

Then $g(x_r) = F(x_{r-1}, y_{r-1}) = g(x_{r-1})$

Therefore $F(x_{r-1}, y_{r-1}) = g(x_{r-1})$

and $g(y_r) = F(y_{r-1}, x_{r-1}) = g(y_{r-1})$

Therefore $F(y_{r-1}, x_{r-1}) = g(y_{r-1})$

That is F and g have a coupled coincidence point.

Now, we assume that $d\big(g(x_n), g(x_{n-1})\big) + d\big(g(y_n), g(y_{n-1})\big) \neq 0$

Since $g(x_{n-1}) \leq g(x_n)$ and $g(y_{n-1}) \geq g(y_n)$, form (2.1.1) and (2.1.2) we have

$$d\big(g(x_{n+1}), g(x_n)\big) + d\big(g(y_{n+1}), g(y_n)\big)$$
$$= d\big(F(x_n, y_n), F(x_{n-1}, y_{n-1})\big) + d\big(F(y_n, x_n)), F(y_{n-1}, x_{n-1})\big)$$

$$\leq \theta\left(d\big(g(x_n),g(x_{n-1})\big),d\big(g(y_n),g(y_{n-1})\big)\right)\alpha\Big(min\big\{d\big(F(x_n,y_n),g(x_n)\big),d\big(F(x_{n-1},y_{n-1}),g(x_n)\big)\big\}$$
$$+\,min\big\{d\big(F(y_n,x_n),g(y_{n-1})\big),d\big(F(y_{n-1},x_{n-1}),g(y_{n-1})\big)\big\}\Big)$$
$$+\,\beta\Big(min\big\{d\big(F(x_n,y_n),g(x_{n-1})\big),d\big(F(x_{n-1},y_{n-1}),g(x_{n-1})\big)\big\}$$
$$+\,min\big\{d\big(F(y_n,x_n),g(y_{n-1})\big),d\big(F(y_{n-1},x_{n-1}),g(y_{n-1})\big)\big\}\Big)$$
$$+\,L\Big(min\big\{d\big(F(x_n,y_n),g(x_{n-1})\big),d\big(F(x_{n-1},y_{n-1}),g(x_n)\big)\big\}$$
$$+\,min\big\{d\big(F(y_n,x_n),g(y_{n-1})\big),d\big(F(y_{n-1},x_{n-1}),g(y_n)\big)\big\}\Big)$$

$$\theta\left(d\big(g(x_n),g(x_{n-1})\big),d\big(g(y_n),g(y_{n-1})\big)\right)\Big[\alpha\Big(min\big\{d\big(g(x_{n+1}),g(x_n)\big),d\big(g(x_n),g(x_n)\big)\big\}$$
$$+\,min\big\{d\big(g(y_{n+1}),g(y_{n-1})\big),d\big(g(y_n),g(y_{n-1})\big)\big\}\Big)$$
$$+\,\beta\Big(min\big\{d\big(g(x_{n+1}),g(x_{n-1})\big),d\big(g(x_n),g(x_{n-1})\big)\big\}$$
$$+\,min\big\{d\big(g(y_{n+1}),g(y_{n-1})\big),d\big(g(y_n),g(y_{n-1})\big)\big\}\Big)$$
$$+\,L\Big(min\big\{d\big(g(x_{n+1}),g(x_{n-1})\big),d\big(g(x_n),g(x_n)\big)\big\}$$
$$+\,min\big\{d\big(g(y_{n+1}),g(y_{n-1})\big),d\big(g(y_n),g(y_n)\big)\big\}\Big)\Big]$$
$$=\theta\left(d\big(g(x_n),g(x_{n-1})\big),d\big(g(y_n),g(y_{n-1})\big)\right)\Big[\alpha\Big(min\big\{d\big(g(y_{n+1}),g(y_{n-1})\big),d\big(g(y_n),g(y_{n-1})\big)\big\}\Big)$$
$$+\,\beta\Big(min\big\{d\big(g(x_{n+1}),g(x_{n-1})\big),d\big(g(y_n),g(x_{n-1})\big)\big\}$$
$$+\,min\big\{d\big(g(y_{n+1}),g(y_{n-1})\big),d\big(g(x_n),g(y_{n-1})\big)\big\}\Big)\Big]$$
$$=\theta\left(d\big(g(x_n),g(x_{n-1})\big),d\big(g(y_n),g(y_{n-1})\big)\right)\Big[\alpha d\big(g(y_n),g(y_{n-1})\big)+\beta d\big(g(x_n),g(x_{n-1})\big)$$
$$+\,\beta d\big(g(y_n),g(y_{n-1})\big)\Big]$$
$$=\theta\left(d\big(g(x_n),g(x_{n-1})\big),d\big(g(y_n),g(y_{n-1})\big)\right)\Big[\alpha d\big(g(y_n),g(y_{n-1})\big)$$
$$+\,\beta d\big(g(x_n),g(x_{n-1})\big)+\beta d\big(g(y_n),g(y_{n-1})\big)\Big]$$
$$=\theta\left(d\big(g(x_n),g(x_{n-1})\big),d\big(g(y_n),g(y_{n-1})\big)\right)\Big[\alpha d\big(g(y_n),g(y_{n-1})\big)$$
$$+\,\beta\left(d\big(g(x_n),g(x_{n-1})\big)+d\big(g(y_n),g(y_{n-1})\big)\right)\Big]$$
$$\leq\Big[(\alpha+\beta)d\big(g(y_n),g(y_{n-1})\big)+\beta d\big(g(x_n),g(x_{n-1})\big)\Big]$$
$$\leq\Big[(\alpha+\beta)d\big(g(y_n),g(y_{n-1})\big)+(\alpha+\beta)d\big(g(x_n),g(x_{n-1})\big)\Big]$$
$$=(\alpha+\beta)\Big[d\big(g(y_n),g(y_{n-1})\big)+d\big(g(x_n),g(x_{n-1})\big)\Big]$$

Set $\rho_n = d\big(g(x_{n+1}),g(x_n)\big)+d\big(g(y_{n+1}),g(y_n)\big)$ and $\delta=(\alpha+\beta)$ then sequence $\{\rho_n\}$ is decreasing as

$$0\leq\rho_n\leq\delta\rho_{n-1}\leq\delta^2\rho_{n-2}\ldots\leq\delta^n\rho_0$$

Which implies

$$\lim_{n\to\infty}\rho_n=\lim_{n\to\infty}d\big(g(x_{n+1}),g(x_n)\big)+d\big(g(y_{n+1}),g(y_n)\big)=0 \qquad (2.1.8)$$

Thus,

$$\lim_{n\to\infty}d\big(g(x_{n+1}),g(x_n)\big)=0$$

and

$$\lim_{n\to\infty} d\big(g(y_{n+1}), g(y_n)\big) = 0$$

We shall prove that $\{g(x_n)\}$ and $\{g(y_n)\}$ are Cauchy sequences

For each $m \geq n$, we have

$$d\big(g(x_m), g(x_n)\big) \leq d\big(g(x_m), g(x_{m-1})\big) + d\big(g(x_{m-1}), g(x_{m-2})\big) + \cdots + d\big(g(x_{n+1}), g(x_n)\big)$$

and

$$d\big(g(y_m), g(y_n)\big) \leq d\big(g(y_m), g(y_{m-1})\big) + d\big(g(y_{m-1}), g(y_{m-2})\big) + \cdots + d\big(g(y_{n+1}), g(y_n)\big)$$

Therefore

$$d\big(g(x_m), g(x_n)\big) + d\big(g(y_m), g(y_n)\big) \leq \rho_{m-1} + \rho_{m-2} + \cdots + \rho_n$$

$$\leq (\delta^{m-1} + \delta^{m-2} + \cdots + \delta^n)\rho_0$$

$$\leq \frac{\delta^n}{1-\delta}\rho_0$$

Which implies that

$$\lim_{n,m\to\infty} d\big(g(x_m), g(x_n)\big) + d\big(g(y_m), g(y_n)\big) = 0 \qquad (2.1.9)$$

This imply that $\{g(x_n)\}$ and $\{g(y_n)\}$ are Cauchy sequence in $g(X)$.

Since $g(x)$ is a complete subspace of X. There exists $(x, y) \in X \times X$ such that $g(x_n) \to g(x)$ and $g(y_n) \to g(y)$.

Since $\{g(x_n)\}$ is a non decreasing sequence $g(x_n) \to g(x)$ and as $\{g(y_n)\}$ is a non increasing sequence $g(y_n) \to g(y)$, by assumption we have $g(x_n) \leq g(x)$ and $g(y_n) \geq g(y)$ for all n.

Since $d(g(x_{n+1}), F(x,y)) + d(g(y_{n+1}), F(y,x)) = d\big(F(x_n, y_n), F(x,y)\big) + d\big(F(y_n, x_n), F(y,x)\big)$

$$\begin{aligned}\leq \theta\Big(d\big(g(x_n), g(y)\big), d\big(g(y_n), g(y)\big)\Big)\big[&\alpha\big(min\{d\big(F(x_n,y_n), g(x_n)\big), d\big(F(x,y), g(x_n)\big)\}\\ &+ min\{d\big(F(y_n,x_n), g(y_n)\big), d\big(F(y,x), g(y)\big)\}\big)\\ &+ \beta\big(min\{d\big(F(x_n,y_n), g(x)\big), d\big(F(x,y), g(x)\big)\}\\ &+ min\{d\big(F(y_n,x_n), g(y)\big), d\big(F(y,x), g(y)\big)\}\big)\\ &+ L\big(min\{d\big(F(x_n,y_n), g(x)\big), d\big(F(x,y), g(x_n)\big)\}\\ &+ min\{d\big(F(y_n,x_n), g(y)\big), d\big(F(y,x), g(y_n)\big)\}\big)\big]\end{aligned}$$

Taking the limit as $n \to \infty$ in the above inequality

We get $d(g(x), F(x,y)) + d(g(y), F(y,x)) = 0$

Hence $g(x) = F(x,y)$ and $g(y) = F(y,x)$

Thus we proved that F and g have a coupled coincidence point.

Theorem 2.2. Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F: X \times X \to X$ and $g: X \to X$ be mappings such that F has the mixed g-monotone property on X such that there exist two elements $x_0, y_0 \in X$ with $g(x_0) \leq F(x_0, y_0)$ and $g(y_0) \geq F(y_0, x_0)$. Suppose there exist $\eta \in \theta$ non-negative real numbers α, β, L with $\alpha + \beta < 1$ such that

$$d\big(F(x,y), F(u,v)\big) + d\big(F(y,x)), F(v,u)\big) \leq$$
$$\theta\Big(d\big(g(x), g(u)\big), d\big(g(y), g(v)\big)\Big)\Big[\alpha\big(min\{d\big(F(x,y), g(x)\big), d\big(F(u,v), g(x)\big)\} +$$
$$min\{d\big(F(y,x), g(v)\big), d\big(F(v,u), g(v)\big)\}\big) + \beta\big(min\{d\big(F(x,y), g(u)\big), d\big(F(u,v), g(u)\big)\} +$$
$$min\{d\big(F(y,x), g(v)\big), d\big(F(v,u), g(v)\big)\}\big) + L\Big(min\{d\big(F(x,y), g(u)\big), d\big(F(u,v), g(x)\big)\} +$$
$$min\Big\{d\Big(F(y,x), \big(g(v)\big)\Big), d\big(F(v,u), g(y)\big)\Big\}\Big)\Big] \quad (2.2.1)$$

for all $(x,y), (u,v) \in X$ with $g(x) \leq g(u)$ and $g(y) \geq g(v)$. Further suppose $F(X \times X) \subseteq g(X)$ and g is continuous non decreasing and commutes with F. and also, suppose either

a. F is continuous or
b. X has the following property
   i. If a non-decreasing sequence $\{x_n\}$ in X converges to $x \in X$ then $x_n \leq x$ for all n.
   ii. If a non -increasing sequence $\{y_n\}$ in X converges to $y \in X$ then $y_n \geq y$ for all n.

Then there exist $x, y \in X$ such that $F(x,y) = g(x)$ and $F(y,x) = g(y)$ that is, F and g have a coupled coincidence point $(x,y) \in X \times X$

Proof: Following the proof of Theorem 2.1we will get two Cauchy sequences $\{g(x_n)\}$ and $\{g(y_n\}$ in X such that $\{g(x_n)\}$ is a non decreasing sequence in X $\{g(y_n)\}$ is a nonincreasing sequence in X. Since X is a complete metric space , there is $(x,y) \in X \times X$ such that $g(x_n) \to x$ and $g(y_n) \to y$. Since g is continuous, we have $g(g(x_n)) \to g(x)$ and $g(g(y_n)) \to g(y)$.

First , suppose that F is continuous,

Then $F(g(x_n), g(y_n)) \to F(x,y)$ and $F(g(y_n), g(x_n)) \to F(y,x)$.

On other hand , we have $F\big(g(x_n), g(y_n)\big) = g\big(F(x_n, y_n)\big) = g(g(x_{n+1})) \to g(x)$ and $F\big(g(y_n), g(x_n)\big) = g\big(F(y_n, x_n)\big) = g(g(y_{n+1})) \to g(y)$.By uniqueness of limit, we get $g(x) = F(x, y)$ and $g(y) = F(y, x)$.

Second, Suppose that (b) holds. Since $\{g(x_n\}$ is a non decreasing sequence in X such that

$g(x_n) \to x$, $\{g(y_n)\}$ is a non decreasing sequence in X such that $g(y_n) \to y$, and g is a nondecreasing function, we get that $g(g(x_n)) \leq g(x)$ and $g(g(y_n)) \geq g(y)$ hold for all $n \in N$ by(2.2.1), we have

$$d\big(g\big(g(x_{n+1})\big), F(x, y)\big) + d\big(g\big(g(y_{n+1})\big), F(y, x)\big)$$
$$= d\left(g\left(F\big(x_{n,}y_n\big)\right), F(x, y)\right) + d\left(g\left(F\big(y_{n,}x_n\big)\right), F(y, x)\right)$$

$$= d\left(\left(F\big(gx_{n,}gy_n\big)\right), F(x, y)\right) + d\left(\left(F\big(gy_{n,}gx_n\big)\right), F(y, x)\right)$$

$$\leq \theta\left(d\left(g\big(g(x_n)\big), g(x)\right), d\left(g\big(g(y_n)\big), g(y)\right)\right)$$
$$\Big[\alpha\left(min\left\{d\left(F(gx_n, gy_n), g\big(g(x_n)\big)\right), d\left(F(x, y), g\big(g(x_n)\big)\right)\right\}\right.$$
$$\left. + min\left\{d\left(F(gy_n, gx_n), g\big(g(y_n)\big)\right), d\big(F(y, x), g(y)\big)\right\}\right)$$
$$+ \beta\big(min\big\{d\big(F(gx_n, gy_n), g(x)\big), d\big(F(x, y), g(x)\big)\big\}$$
$$+ min\big\{d\big(F(gy_n, gx_n), g(y)\big), d\big(F(y, x), g(y)\big)\big\}\big)$$
$$+ L\left(min\left\{d\big(F(gx_n, gy_n), g(x)\big), d\left(F(x, y), g\big(g(x_n)\big)\right)\right\}\right.$$
$$\left. + min\left\{d\big(F(gy_n, gx_n), g(y)\big), d\left(F(y, x), g\big(g(y_n)\big)\right)\right\}\right)\Big]$$

$$\leq \theta\left(d\left(g\big(g(x_n)\big), g(x)\right), d\left(g\big(g(y_n)\big), g(y)\right)\right)$$
$$\Big[\alpha\left(min\left\{d\left(g\big(g(x_{n+1})\big), g\big(g(x_n)\big)\right), d\left(F(x, y), g\big(g(x_n)\big)\right)\right\}\right.$$
$$\left. + min\left\{d\left(g\big(g(y_{n+1})\big), g\big(g(y_n)\big)\right), d\big(F(y, x), g(y)\big)\right\}\right)$$
$$+ \beta\left(min\left\{d\left(g\big(g(x_{n+1})\big), g(x)\right), d\big(F(x, y), g(x)\big)\right\}\right.$$
$$\left. + min\left\{d\left(g\big(g(y_{n+1})\big), g(y)\right), d\big(F(y, x), g(y)\big)\right\}\right)$$
$$+ L\left(min\left\{d\left(g\big(g(x_{n+1})\big), g(x)\right), d\left(F(x, y), g\big(g(x_n)\big)\right)\right\}\right.$$
$$\left. + min\left\{d\left(g\big(g(y_{n+1})\big), g(y)\right), d\left(F(y, x), g\big(g(y_n)\big)\right)\right\}\right)\Big]$$

Letting $n \to +\infty$ we get $d(g(x), F(x, y)) + d(g(y), F(y, x)) = 0$ and

hence $g(x) = F(x, y)$and $g(y) = F(y, x)$

Thus we proved that F and g have a coupled coincidences point.

Theorem 2.3:

In addition to the hypotheses of theorem 2.1. Suppose that $L = 0$ and for every $(x,y),(x^*,y^*) \in X \times X$ there exists a $(u,v) \in X \times X$ such that $\big(F(u,v),F(v,u)\big)$ is comparable to $\big(F(x,y),F(y,x)\big)$ and $\big(F(x^*,y^*),F(y^*,x^*)\big)$.Then F and g have a unique coupled common fixed point, that is there exists a unique $(x,y) \in X \times X$ such that $x = g(x) = F(x,y)$ and $y = g(y) = F(y,x)$.

Proof:

From Theorem 2.1 the set of coupled coincidence points of F and g is non-empty. Suppose (x,y) and (x^*,y^*) are coupled coincidence points of F, that is $g(x) = F(x,y)$ and $g(y) = F(y,x)$,

$g(x^*) = F(x^*,y^*)$ and $g(y^*) = F(y^*,x^*)$ then $g(x) = g(x^*)$ and $g(y) = g(y^*)$.
(2.3.1)

By assumption, there exists $(u,v) \in X \times X$ such that $\big(F(u,v),F(v,u)\big)$ is comparable to $\big(F(x,y),F(y,x)\big)$ and $\big(F(x^*,y^*),F(y^*,x^*)\big)$. Put $u_0 = u$, $v_0 = v$ and choose u_1, $v_1 \in X$ so that

$g(u_1) = F(u_0,v_0)$ and $g(v_1) = F(v_0,u_0)$ then similarly as in the proof of Theorem 2.1, we can inductively define sequence $\{g(u_n)\}$, $\{g(v_n)\}$ such that $g(u_{n+1}) = F(u_n,v_n)$ and $g(v_{n+1}) = F(v_n,u_n)$ for all n.

Further , set $x_0 = x, y_0 = y, {x^*}_0 = x^*, {y^*}_0 = y^*$ and , on the same way, define the sequence $\{g(x_n)\},\{g(y_n)\}$ and $\{g({x^*}_n)\},\{g({y^*}_n)\}$. Then it is easy to show that

$g(x_n) \to F(x,y)$

$g(y_n) \to F(y,x)$

$g({x^*}_n) \to F(x^*,y^*)$

$g({y^*}_n) \to F(y^*,x^*)$ for all $n \geq 1$

$$\big(F(x,y),F(y,x)\big) = \big(F(x_0,y_0),F(y_0,x_0)\big) = \big(g(x_1),g(y_1)\big) = \big(g(x),g(y)\big)$$

and

$\big(F(u,v),F(v,u)\big) = \big(g(u_1),g(v_1)\big)$ are comparable, then $g(x) \leq g(u_1)$ and $g(y) \geq g(v_1)$ it is easy to show that$\big(g(x),g(y)\big)$ $\big(g(u_n),g(v_n)\big)$ are comparable that is $g(x) \leq g(u_n)$ and $g(y) \geq g(v_n)$ for all $n \geq 1$ thus from(2.1.1) , we have

$$d\big(g(x),g(u_{n+1})\big) + d\big(g(y),g(v_{n+1})\big) = d\big(F(x,y),F(u_n,v_n)\big) + d\big(F(y,x),F(v_n,u_n)\big)$$

$$\leq \theta\left(d(g(x), g(u_n)), d(g(y), g(v_n))\right)$$
$$[\alpha(min\{d(F(x,y), g(x)), d(F(u_n, v_n), g(x))\} + min\{d(F(y,x), g(v_n)), d(F(v_n, u_n), g(v_n))\}) + \beta(min\{d(F(x,y), g(u_n)), d(F(v_n, u_n), g(u_n))\} + min\{d(F(y,x), g(v_n)), d(F(v_n, u_n), g(v_n))\})]$$
$$\leq \theta\left(d(g(x), g(u_n)), d(g(y), g(v_n))\right)[\alpha(min\{d(g(x), g(x)), d(F(u_n, v_n), g(x))\} + min\{d(g(y), g(v_n)), d(F(v_n, u_n), g(v_n))\}) + \beta(min\{d(g(x), g(u_n)), d(F(v_n, u_n), g(u_n))\} + min\{d(g(y), g(v_n)), d(F(v_n, u_n), g(v_n))\})]$$

$$= (\alpha+\beta)d(g(y), g(v_n)) + \beta d(g(x), g(u_n))$$

$$\leq (\alpha+\beta)d(g(y), g(v_n)) + (\alpha+\beta)d(g(x), g(u_n))$$

$$\leq (\alpha+\beta)[d(g(x), g(u_n)) + d(g(y), g(v_n))]$$

$$\leq (\alpha+\beta)^2[d(g(x), g(u_n)) + d(g(y), g(v_n))]$$

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$$\leq (\alpha+\beta)^{n+1}[d(g(x), g(u_n)) + d(g(y), g(v_n))]$$

Taking the limit as $n \to \infty$ we get

$$\lim_{n\to\infty} d(g(x), g(u_n)) + d(g(y), g(v_n)) = 0$$

$$\lim_{n\to\infty} d(g(x), g(u_n)) = 0 \quad , \quad \lim_{n\to\infty} d(g(y), g(v_n)) = 0 \qquad (2.3.2)$$

Similarly, we show that

$$\lim_{n\to\infty} d(g(x^*), g(u_n)) = 0 \quad , \quad \lim_{n\to\infty} d(g(y^*), g(v_n)) = 0 \qquad (2.3.3)$$

By the triangle inequality (2.3.2) and (2.3.3)

$d(g(x), g(x^*)) \leq d(g(x), g(u_{n+1})) + d(g(x^*), g(u_{n+1})) \to 0$ as $n \to \infty$

$d(g(y), g(y^*)) \leq d(g(y), g(v_{n+1})) + d(g(y^*), g(v_{n+1})) \to 0$ as $n \to \infty$

we have $g(x) = g(x^*)$ and $g(y) = g(y^*)$

This implies that $d(g(x), g(y)) = d(g(x^*), g(y^*))$

Since $g(x) = F(x,y)$ and $g(y) = F(y,x)$

$g\big(g(x)\big) = g(F(x,y)) = F(g(x),g(y))$ and $g\big(g(y)\big) = g(F(y,x)) = F(g(y),g(x))$ (2.3.4)

Denote $g(x) = z$ and $g(y) = w$

Then $g(z) = F(z,w)$ and $g(w) = F(w,z)$ (2.3.4)

Thus (z,w) is a coupled coincidence point.

Then from (2.3.3) with $x^* = z$ and $y^* = w$ it follows $g(z) = g(x)$ and $g(w) = g(y)$.

That is $g(z) = z$ and $g(w) = w$. (2.3.5)

From(2.3.4) and (2.3.5)

$z = g(z) = F(z,w)$ and $w = g(w) = F(w,z)$

Therefore (z,w) is a coupled common fixed point of F and g.

To prove the uniqueness, assume that (p,q) is another coupled common fixed point .

Then by (2.3.4) we have $p = g(p) = g(z) = z$ and $q = g(q) = g(w) = w$.

Therefore (z,w) is a unique coupled common fixed point of F and g.

Theorem 2.4: In addition to hypotheses of Theorem 2.1, if gx_0 and gy_0 are comparable and $L = 0$, then F and g have a coupled coincidence point (x,y) such that $gx = F(x,y) = F(y,x) = gy$.

Proof: By Theorem 2.1 we construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $gx_n \to gx$ and $gy_n \to gy$ where (x,y) coincidence point of F and g. Suppose $gx_0 \leq gy_0$, then inductively

$gx_n \leq gy_n$ and for all $n \in N \cup \{0\}$.

Thus, by (2.1.1) we have

$$d(gx_n, gy_n) + d(gy_n, gx_n) = d(F(x_{n-1},y_{n-1}),F(y_{n-1},x_{n-1})) + d(F(y_{n-1},x_{n-1}),F(x_{n-1},y_{n-1}))$$

$$\leq \theta\Big(d\big(g(x_{n-1}),g(y_{n-1})\big),d\big(g(y_{n-1}),g(x_{n-1})\big)\Big)$$
$$[\alpha(min\{d(F(x_{n-1},y_{n-1}),g(x_{n-1})),d(F(y_{n-1},x_{n-1}),g(x_{n-1}))\}$$
$$+ min\{d(F(y_{n-1},x_{n-1}),g(x_{n-1})),d(F(x_{n-1},y_{n-1}),g(x_{n-1}))\})$$
$$+ \beta(min\{d(F(x_{n-1},y_{n-1}),g(y_{n-1})),d(F(y_{n-1},x_{n-1}),g(y_{n-1}))\}$$
$$+ min\{d(F(y_{n-1},x_{n-1}),g(x_{n-1})),d(F(x_{n-1},y_{n-1}),g(x_{n-1}))\})]$$

$$\leq \theta\left(d\big(g(x_{n-1}),g(y_{n-1})\big),d\big(g(y_{n-1}),g(x_{n-1})\big)\right)[\alpha(min\{d(g(x_n),g(x_{n-1})),d(g(y_n),g(x_{n-1}))\}$$
$$+ min\{d(g(y_n),g(x_{n-1})),d(g(x_n),g(x_{n-1}))\})$$
$$+ \beta(min\{d(g(x_n),g(y_{n-1})),d(g(y_n),g(y_{n-1}))\}$$
$$+ min\{d(g(y_n),g(x_{n-1})),d(g(x_n),g(y_{n-1}))\})]$$

Taking the limit as $n \to +\infty$ we get

$$d(gx,gy) + d(gy,gx) \leq \beta d(gx,gy)$$

$$2d(gx,gy) \leq \beta d(gx,gy)$$

$$d(gx,gy) = 0$$

$$gx = gy$$

Hence $F(x,y) = gx = gy = F(y,x)$

A similar argument can be used if $gy_0 \leq gx_0$.

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