INNOVATIVE CLASS OF LAPLACE TRANSFORM OF CONVOLUTION-TYPE INTEGRALS INVOLVING GENERALIZED HYPERGEOMETRIC FUNCTIONS

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Abstract

The Laplace transform of convolution-type integrals involving generalized hypergeometric functions is an advanced mathematical concept that combines the principles of Laplace transforms and special functions, particularly generalized hypergeometric functions (Exton 1976, 1978). These integrals often arise in the study of differential and integral equations and can be evaluated using classical summation theorems and properties of generalized hypergeometric functions (Andrew, 1985). Notable examples include Kim et al. (2011), Milovanović et al. (2018, 2019), and Rathie et al. (2021). Our objective is to establish the Laplace transform of a convolution-type integral with the help of generalized hypergeometric formulas established by Masjed-Jamei and Koepf (2018).

Key words: Generalized hypergeometric function, Classical summation theorems Generalization, Laplace Transform, Convolution-type integrals.

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1. Introduction

Hypergeometric functions occupy a central role within the broader class of special functions, serving as a unifying framework across diverse branches of mathematics and science. Their importance is underscored by the fact that many fundamental functions—such as trigonometric, exponential, and logarithmic functions, as well as Bessel functions, Whittaker functions, error functions, and classical orthogonal polynomials—can all be expressed as particular instances of the generalized hypergeometric series. This unifying characteristic makes hypergeometric functions a powerful tool in both theoretical investigations and applied analyses, spanning fields from mathematical physics to computational methods and beyond.

The generalized hypergeometric function is defined as follows:

$$ {}_pF_q\left[\begin{matrix} a_1, a_2, ..., a_p \\ b_1, b_2, ..., b_q \end{matrix} ; \text{z}\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n ... (a_p)_n}{(b_1)_n (b_2)_n ... (b_p)_n} \frac{z^n}{n!} \;\; = {}_pF_q\left[\begin{matrix} (a_p)_n \\ (b_q)_n \end{matrix} ; \text{z}\right]. \quad (1.1) $$

Where

$$ (a)_n = \begin{cases} 1, & n=0 \\ (a)(a+1)(a+2).......(a+n-1), & n \in N \end{cases} \quad (1.2) $$

Is known as Pochhammer's symbol or shifted factorial.
In addition, in terms of Gamma function shifted factorial is

$$ (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}, \quad (a)_0 = 1, \quad a \neq 0, -1, -2... \quad (1.3) $$

Also, p and q are either zero or positive integers and the argument z may take any real or complex value, provided none of the denominator parameters (b_q) in Eq. (1.1) is zero or negative integer. This generalized hypergeometric function is convergent or divergent with the following restrictions:

(i) If $p \leq q$, the series is convergent for all finite z.
(ii) If $p = q + 1$, the series is convergent for $|z| < 1$ and diverges for $|z| > 1$.
(iii) If $p > q + 1$, the series diverges for $z \neq 0$.

If $p = q + 1$, the series is absolutely convergent on the circle $|z| = 1$ if

$$ \Re\left[\sum_{j=0}^{q} b_j - \sum_{j=0}^{q+1} a_j\right] > 0 $$

For specific parameter values p and q, Eq. (1.1) can be reduced to expressions involving Gamma functions, thereby enhancing its applicability. In such cases, the associated hypergeometric functions are characterized by the classical summation theorems. Classical summation theorems provide a powerful framework for working with generalized hypergeometric functions and contribute to the understanding and application of these functions across diverse areas of science and mathematics. Only a few summation theorems are available in the literature, and it is well known that the classical summation theorems such as of Gauss, Gauss's second, Kummer's, and Bailey for the series ${}_2F_1$; Watson, Dixon, and Whipple for the series ${}_3F_2$ play an important role in the theory of generalized hypergeometric series. Here, we will mention a few of the described famous summation theorems that will apply to our current research (Slater 1966; Rainville 1967):

1. Gauss's summation theorem:

$$ {}_2F_1\left[\begin{matrix} a,b \\ c \end{matrix} ; 1\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \tag{1.4}$$

provided $\Re(c-a-b) > 0.$

2. Kummer's theorem:

$$ {}_2F_1\left[\begin{matrix} a,b \\ 1+a-b \end{matrix} ; -1\right] = \frac{\Gamma(1+a-b)\Gamma\left(1+\frac{a}{2}\right)}{\Gamma\left(1-b+\frac{a}{2}\right)\Gamma(1+a)}.. \tag{1.5}$$

Provided $\Re(b) < 1$

3. Gauss's Second summation theorem:

$$ {}_2F_1\left[\begin{matrix} a,b \\ \frac{1}{2}(1+a+b) \end{matrix} ; \frac{1}{2}\right] = \frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}(a+b+1)\right)}{\Gamma\left(\frac{1}{2}(a+1)\right)\Gamma\left(\frac{1}{2}(b+1)\right)}.. \tag{1.6}$$

4. Bailey's summation theorem:

$$ {}_2F_1\left[\begin{matrix} a,1-a \\ b \end{matrix} ; \frac{1}{2}\right] = \frac{\Gamma\left(\frac{1}{2}b\right)\Gamma\left(\frac{1}{2}(b+1)\right)}{\Gamma\left(\frac{1}{2}(a+b)\right)\Gamma\left(\frac{1}{2}(b-a+1)\right)}. \tag{1.7}$$

In the vast realm of mathematical research, the study of Laplace transforms and their applications remains an ongoing pursuit. The Laplace transform of convolution-type integrals serves as a powerful tool for solving a variety of problems in mathematics, physics, engineering, and statistics. This tool is widely utilized in these fields, and it continues to be an active area of research.

A convolution integral, often denoted by "*" in mathematical operations that combines two functions to produce a third function. It is defined as follows for two functions $f(t)$ and $g(t)$ (Sneddon 1979):

$$ (f*g) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau. \tag{1.8}$$

In the formula, the function $f(t)$ is flipped and shifted by $g(t)$, and the area under the overlap is computed, resulting a new function. The convolution operation is commutative, meaning $f*g = g*f$, and it is often used to model the response of a linear time invariant system to an input signal.

A sophisticated mathematical idea that combines the ideas of Laplace transforms with special functions—specifically, generalized hypergeometric functions—is the Laplace transform of convolution-type integrals. The general product theorem for the Laplace transform is (Slater 1960) as follows:

$$g_1(t)g_2(t)=\int_0^\infty\left\{\int_0^\tau f_1(\tau)f_2(t-\tau)d\tau\right\}e^{-st}dt. \quad (1.9)$$

The Laplace transform of convolution-type integrals utilizing generalized hypergeometric functions enables investigation of systems and phenomena represented by these special functions,we started by introducing the concepts of Laplace transform and convolution type integral. The general product theorem for the Laplace transform is (Slater 1960) as follows:

$$g_1(t)g_2(t)=\int_0^\infty\left\{\int_0^\tau f_1(\tau)f_2(t-\tau)d\tau\right\}e^{-st}dt. \quad (1.10)$$

This theorem was applied to a generalized hypergeometric function which gives the following result:

$$\begin{aligned}&\int_0^\infty e^{-st}\left\{\int_0^t \tau^{\mu-1}{}_AF_B\left[(a);(b);\kappa\tau\right](t-\tau)^{\nu-1}{}_{A'}F_{B'}\left[(a');(b');\kappa'(t-\tau)\right]d\tau\right\}dt\\&=\Gamma(\mu)\Gamma(\nu)s^{\mu-\nu}{}_{A+1}F_B\left[(a),\mu;(b);\kappa/s\right]{}_{A'+1}F_{B'}\left[(a'),\nu;(b');\kappa'/s\right].\end{aligned} \quad (1.11)$$

Where $A<B,\ A'<B'$,$\Re(\mu)>0,\Re(\nu)>0$ and $\Re(s)>0$.

Again For $A=B=A'=B'=1$ and $\Re(\mu)>0,\Re(\nu)>0,\Re(s)>\Re(\kappa)$ we have (Slater, 1960):

$$\begin{aligned}&\int_0^\infty e^{-st}\left\{\int_0^t \tau^{\mu-1}{}_AF_B\left[(a);(b);\kappa\tau\right](t-\tau)^{\nu-1}{}_{A'}F_{B'}\left[(a');(b');\kappa'(t-\tau)\right]d\tau\right\}dt\\&=\Gamma(\mu)\Gamma(\nu)s^{\mu-\nu}{}_{A+1}F_B\left[(a),\mu;(b);\kappa/s\right]{}_{A'+1}F_{B'}\left[(a'),\nu;(b');\kappa'/s\right].\end{aligned} \quad (1.12)$$

In a further section, we have established the Laplace transform of a Convolution-type integral involving the product of a generalization of classical summation theorems for the series ${}_2F_1$ given by Masjed-Jamei and Koepf (2018a).

2. Methodology and Results Required:

We establish the Laplace transform of convolution-type integrals involving the series ${}_3\mathrm{F}_3$ by employing the generalization of the classical summation theorem for the series ${}_2\mathrm{F}_1$ established by Masjed-Jamei and Koepf (2018a).

We aim to provide as many Laplace transforms of convolution-type related to the product of the series ${}_3\mathrm{F}_3$ Relevant techniques and ideas from the cited research will be incorporated. The following are the results we used to establish our findings:

Generalizations of various classical summation theorems for the series ${}_2F_1$, ${}_3F_2$, ${}_4F_3$, ${}_5F_4$, and ${}_6F_5$ given by Masjed-Jamei and Koepf (2018) are stated below:
The main relation established in the paper is:

$$
{}_pF_q\left[\begin{matrix} a_1,a_2,...,a_{p-1},1 \\ b_1,b_2,...,b_{q-1},m \end{matrix};\mathrm{z}\right] = \frac{\Gamma(b_1)...\Gamma(b_{q-1})\Gamma(a_1-m+1)...\Gamma(a_{p-1}-m+1)}{\Gamma(a_1)...\Gamma(a_{p-1})\Gamma(b_1-m+1)...\Gamma(b_{q-1}-m+1)}\frac{(m-1)!}{z^{m-1}}
\times\left({}_{p-1}F_{q-1}\left[\begin{matrix} a_1-m+1,...,a_{p-1}-m+1 \\ b_1-m+1,...,b_{q-1}-m+1 \end{matrix};\mathrm{z}\right] - {}_{p-1}\overset{(m-2)}{F}{}_{q-1}\left[\begin{matrix} a_1-m+1,...,a_{p-1}-m+1 \\ b_1-m+1,...,b_{q-1}-m+1 \end{matrix};\mathrm{z}\right]\right). \tag{2.1}
$$

Using the above relation, various special cases that generalize all the classical summation theorems as mentioned below were investigated:

- When p = 3, q = 2 and z = 1, Eq. (2.1) is simplified as:

$$
{}_3F_2\left[\begin{matrix} a,b,1 \\ c,m \end{matrix};1\right] = \frac{\Gamma(m)\Gamma(c)\Gamma(a-m+1)\Gamma(b-m+1)}{\Gamma(a)\Gamma(b)\Gamma(c-m+1)}
\times\left(\frac{\Gamma(c-m+1)\Gamma(c-a-b+m-1)}{\Gamma(c-a)\Gamma(c-b)} - {}_2\overset{(m-2)}{F}{}_1\left[\begin{matrix} a-m+1,b-m+1 \\ c-m+1 \end{matrix};1\right]\right)=\Omega_1. \tag{2.2}
$$

Eq. (2.2) gives a generalization of Gauss's summation theorem, Eq. (1.4).

- When p = 3, q = 2 and z = -1, Eq. (2.1) is simplified as:

$$
{}_3F_2\left[\begin{matrix} a,b,1 \\ a-b+m,m \end{matrix};-1\right] = (-1)^{m-1}\frac{\Gamma(m)\,\Gamma(a-b+m)\,\Gamma(a-m+1)\,\Gamma(b-m+1)}{\Gamma(a)\,\Gamma(b)\,\Gamma(a-b+1)}
\times\left(\frac{\Gamma(a-b+1)\Gamma\left(1+\frac{a-m+1}{2}\right)}{\Gamma(2+a-m)\Gamma\left(m-b+\frac{a-m+1}{2}\right)} - {}_2\overset{(m-2)}{F}{}_1\left[\begin{matrix} a-m+1,b-m+1 \\ c-m+1 \end{matrix};-1\right]\right)=\Omega_2. \tag{2.3}
$$

Eq. (2.3) gives a generalization of Kummer's summation theorem, Eq. (1.5).

- When p = 3, q = 2 and z = ½, Eq. (2.1) is simplified as:

$$
{}_3F_2\left[\begin{matrix} a,b,1 \\ \frac{a+b+1}{2},m \end{matrix};\frac{1}{2}\right] = (2)^{m-1}\frac{\Gamma(m)\Gamma\left(\frac{a+b+1}{2}\right)\Gamma(a-m+1)\Gamma(b-m+1)}{\Gamma(a)\Gamma(b)\Gamma\left(-m+\frac{a+b+3}{2}\right)}
\times\left(\frac{\sqrt{\pi}\,\Gamma\left(-m+\frac{a+b+3}{2}\right)}{\Gamma\left(1+\frac{a-m}{2}\right)\Gamma\left(1+\frac{b-m}{2}\right)} - {}_2\overset{(m-2)}{F}{}_1\left[\begin{matrix} a-m+1,b-m+1 \\ -m+1+\frac{a+b+1}{2} \end{matrix};\frac{1}{2}\right]\right)=\Omega_3 \tag{2.4}
$$

Eq. (2.4) gives a generalization of Gauss's second summation theorem, Eq. (1.5).

- When p = 3, q = 2 and z = ½, Eq. (2.1) is simplified as:

$$
\begin{aligned}
&{}_3F_2\left[\begin{matrix} a, 2m-a-1, 1 \\ b, m \end{matrix} ; \frac{1}{2}\right] \\
&=(2)^{m-1}\frac{\Gamma(m)\,\Gamma(b)\,\Gamma(a-m+1)\,\Gamma(m-a)}{\Gamma(a)\,\Gamma(2m-a-1)\,\Gamma(b-m+1)} \qquad (2.5)\\
&\times\left(\frac{\Gamma\left(\frac{b-m+1}{2}\right)\Gamma\left(\frac{b-m+2}{2}\right)}{\Gamma\left(-m+1+\frac{a+b}{2}\right)\Gamma\left(\frac{b-a+1}{2}\right)}-{}_2^{(m-2)}F_1\left[\begin{matrix} a-m+1, m-a \\ b-m+1 \end{matrix} ; \frac{1}{2}\right]\right)=\Omega_4.
\end{aligned}
$$

Eq. (2.5) gives a generalization of Bailey's summation theorem, Eq. (1.5).

Building upon the results presented in Eq. (2.2) to Eq. (2.5), along with the classical summation theorems, we have derived a novel class of Laplace transforms involving convolution-type integrals. These transforms not only extend the existing theoretical framework but also offer significant potential for practical application. In particular, they provide analytical tools for solving a wide range of differential and integral equations commonly encountered in engineering, physics, and mathematical physics.

3. Laplace Transform of Convolution-Type Integrals Involving the Product of Two ${}_3F_3$ series

4.

Theorem 3.1: For $\Re(d) > 0$, $\Re(d') > 0$ and $\Re(s) > 0$, following result holds true:

$$
\begin{aligned}
&\int_0^\infty e^{-st}\left\{\int_0^t \tau^{d-1}\,{}_3F_3\left[\begin{matrix} a, b, 1 \\ c, d, m \end{matrix} ; \tau s\right](t-\tau)^{d'-1}\,{}_3F_3\left[\begin{matrix} a', b', 1 \\ c', d', m \end{matrix} ; (t-\tau)s\right]d\tau\right\}dt \qquad (3.1)\\
&=\frac{\Gamma(d)\Gamma(d')}{s^{d+d'}}\Omega_1(a,b,c)\,\Omega_1(a',b',c').
\end{aligned}
$$

Where Ω_1 is given by Eq. (2.2).

Proof: For proving Theorem 3.1, denoting the right-hand side of Theorem 3.1 by, I and expressing ${}_3F_3$ as a series:

$$
I=\int_0^\infty e^{-st}\left\{\int_0^t \tau^{d+n-1}(t-\tau)^{d+k'-1}\sum_{n=0}^\infty\frac{(a)_n(b)_n(1)_n s^n}{(c)_n(d)_n(m)_n n!}\sum_{k=0}^\infty\frac{(a')_k(b')_k(1')_k s^k}{(c')_k(d')_k(m)_k k!}d\tau\right\}dt,
$$

By changing the order of integration and summation

$$
I=\sum_{n=0}^\infty\frac{(a)_n(b)_n(1)_n s^n}{(c)_n(d)_n(m)_n n!}\times\sum_{k=0}^\infty\frac{(a')_k(b')_k(1')_k s^k}{(c')_k(d')_k(m)_k k!}\int_0^\infty e^{-st}\left\{\int_0^t \tau^{d+n-1}(t-\tau)^{d+k'-1}d\tau\right\}dt,
$$

Now using the definition of the general product theorem for the Laplace transform Eq. (1.7) with Eq. (1.8), we have

$$I=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n(1)_n s^n}{(c)_n(d)_n(m)_n n!}\times\sum_{k=0}^{\infty}\frac{(a')_k(b')_k(1')_k s^k}{(c')_k(d')_k(m)_k k!}\times\frac{\Gamma(d+n)}{s^{d+n}}\times\frac{\Gamma(d'+k)}{s^{d'+k}},$$

Employing Eq. (1.3) and summing up the series, we have

$$=\frac{\Gamma(d)\Gamma(d')}{s^{d+d'}}\,{}_3F_2\left[\begin{matrix}a,b,1\\c,m\end{matrix};1\right]{}_3F_2\left[\begin{matrix}a',b',1\\c',m\end{matrix};1\right]$$

Now, we observe that Eq. (2.3) can be employed to asses ${}_3F_2$, and we reach to right side of Eq. (3.1).

This completes the proof of Theorem 3.1.

Corollary 1. If we take $m=1,2,3$ in Theorem 3.1, we respectively get the following integrals:

$$\int_0^{\infty}e^{-st}\left\{\int_0^{t}\tau^{d-1}\,{}_2F_2\left[\begin{matrix}a,b\\c,d\end{matrix};\tau s\right](t-\tau)^{d'-1}\,{}_2F_2\left[\begin{matrix}a',b'\\c',d'\end{matrix};(t-\tau)s\right]d\tau\right\}dt$$

$$=\frac{\Gamma(d)\Gamma(d')}{s^{d+d'}}\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\frac{\Gamma(c')\Gamma(c'-a'-b')}{\Gamma(c'-a')\Gamma(c'-b')},$$

$$\int_0^{\infty}e^{-st}\left\{\int_0^{t}\tau^{d-1}\,{}_3F_3\left[\begin{matrix}a,b,1\\c,d,2\end{matrix};\tau s\right](t-\tau)^{d'-1}\,{}_3F_3\left[\begin{matrix}a',b',1\\c',d',2\end{matrix};(t-\tau)s\right]d\tau\right\}dt$$

$$=\frac{\Gamma(d)\Gamma(d')}{s^{d+d'}}\left(\frac{(c-1)(c'-1)}{(a-1)(b-1)(a'-1)(b'-1)}\right)\left(\frac{\Gamma(c-1)\Gamma(c-a-b+1)}{\Gamma(c-a)\Gamma(c-b)}-1\right)$$

$$\times\left(\frac{\Gamma(c'-1)\Gamma(c'-a'-b'+1)}{\Gamma(c'-a')\Gamma(c'-b')}-1\right),$$

and

$$\int_0^\infty e^{-st}\left\{\int_0^t \tau^{d-1}\,{}_3F_3\left[\begin{matrix}a,b,1\\c,d,3\end{matrix};\ \tau s\right](t-\tau)^{d'-1}\,{}_3F_3\left[\begin{matrix}a',b',1\\c',d',3\end{matrix};\ (t-\tau)s\right]d\tau\right\}dt$$
$$=\frac{\Gamma(d)\Gamma(d')}{s^{d+d'}}\left(\frac{2(c-2)_2(c'-2)_2}{(a-2)_2(b-2)_2(a'-2)_2(b'-2)}\right)$$
$$\times\left(\frac{\Gamma(c-2)\Gamma(c-a-b+2)}{\Gamma(c-a)\Gamma(c-b)}-\frac{ab+c-2a-2b+2}{c-2}\right)$$
$$\times\left(\frac{\Gamma(c'-2)\Gamma(c'-a'-b'+2)}{\Gamma(c'-a')\Gamma(c'-b')}-\frac{a'b'+c'-2a'-2b'+2}{c'-2}\right).$$

Remark: *The same proof techniques is applied to establish the proof of the following theorems, so we are omitting the proofs.*

Theorem 3.2: For $\Re(d)>0$, $\Re(d')>0$ and $\Re(s)>0$, following result holds true:

$$\int_0^\infty e^{-st}\left\{\int_0^t \tau^{d-1}\,{}_3F_3\left[\begin{matrix}a,b,1\\c,d,m\end{matrix};\ \tau s\right](t-\tau)^{d'-1}\,{}_3F_3\left[\begin{matrix}a',b',1\\a'-b'+m,d',m\end{matrix};\ -(t-\tau)s\right]d\tau\right\}dt \tag{3.2}$$
$$=\frac{\Gamma(d)\Gamma(d')}{s^{d+d'}}\Omega_1(a,b,c)\,\Omega_2(a',b',a'-b'+m).$$

Where Ω_1 and Ω_2 is given by Eq. (2.2) and Eq. (2.3) respectively.

Corollary 2. If we take $m=1,2,3$ in Theorem 3.2, we respectively get the following new integrals:

$$\int_0^\infty e^{-st}\left\{\int_0^t \tau^{d-1}\,{}_2F_2\left[\begin{matrix}a,b\\c,d\end{matrix};\ \tau s\right](t-\tau)^{d'-1}\,{}_2F_2\left[\begin{matrix}a',b'\\1+a'-b',d'\end{matrix};\ -(t-\tau)s\right]d\tau\right\}dt$$
$$=\frac{\Gamma(d)\Gamma(d')}{s^{d+d'}}\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\frac{\Gamma(1+a'-b')\Gamma\left(1+\frac{a'}{2}\right)}{\Gamma\left(1-b'+\frac{a'}{2}\right)\Gamma(1+a')},$$

$$\int_0^\infty e^{-st}\left\{\int_0^t \tau^{d-1}\,{}_2F_2\left[\begin{matrix}a,b\\c,d\end{matrix};\tau s\right](t-\tau)^{d'-1}\,{}_2F_2\left[\begin{matrix}a',b'\\1+a'-b',d'\end{matrix};-(t-\tau)s\right]d\tau\right\}dt$$

$$=\frac{\Gamma(d)\Gamma(d')}{s^{d+d'}}\frac{(c-1)}{(a-1)(b-1)}\frac{(a'-b'+1)}{(a'-1)(b'-1)}$$

$$\times\left\{\frac{\Gamma(c-1)\Gamma(c-a-b+1)}{\Gamma(c-a)\Gamma(c-b)}-1\right\}\left\{1-\frac{\Gamma(1+a'-b')\Gamma\left(\frac{a'}{2}+\frac{1}{2}\right)}{\Gamma(a')\Gamma\left(\frac{a'}{2}-b'+\frac{3}{2}\right)}\right\},$$

and

$$\int_0^\infty e^{-st}\left\{\int_0^t \tau^{d-1}\,{}_3F_3\left[\begin{matrix}a,b,1\\c,d,3\end{matrix};\tau s\right](t-\tau)^{d'-1}\,{}_3F_3\left[\begin{matrix}a',b',1\\a'-b'+3,d',3\end{matrix};-(t-\tau)s\right]d\tau\right\}dt$$

$$=\frac{\Gamma(d)\Gamma(d')}{s^{d+d'}}\frac{4(c-2)_2}{(a-2)_2(b-2)_2}\left\{\frac{\Gamma(c-2)\Gamma(c-a-b+2)}{\Gamma(c-a)\Gamma(c-b)}-\frac{ab+c-2a-2b+2}{c-2}\right\}$$

$$\times\frac{(a'-b'+1)_2}{(a'-2)_2(b'-2)_2}\left\{\frac{\Gamma(a'-b'+1)\Gamma\left(\frac{a'}{2}\right)}{\Gamma(a'-1)\Gamma\left(-b'+2+\frac{a'}{2}\right)}-\frac{3a'-b'-a'b'-3}{a'-b'+1}\right\}.$$

Theorem 3.3: For $\Re(d)>0$, $\Re(d')>0$ and $\Re(s)>0$, following result holds true:

$$\int_0^\infty e^{-st}\left\{\int_0^t \tau^{d-1}\,{}_3F_3\left[\begin{matrix}a,b,1\\c,d,m\end{matrix};\tau s\right](t-\tau)^{d'-1}\,{}_3F_3\left[\begin{matrix}a',b',1\\(a'+b'+1)/2,d',m\end{matrix};\frac{1}{2}(t-\tau)s\right]d\tau\right\}dt$$

$$=\frac{\Gamma(d)\Gamma(d')}{s^{d+d'}}\Omega_1(a,b,c)\,\Omega_3(a',b',(a'+b'+1)/2).$$

(3.3)

Where Ω_1 and Ω_3 is given by Eq. (2.2) and Eq. (2.4) respectively.

Corollary 3. If we take $m=1,2,3$ in Theorem 3.3. We respectively get the following new integrals:

$$\int_0^\infty e^{-st}\left\{\int_0^t \tau^{d-1}{}_2F_2\left[\begin{matrix}a,b\\c,d\end{matrix};\tau s\right](t-\tau)^{d'-1}{}_2F_2\left[\begin{matrix}a',b'\\(a'+b'+1)/2,d'\end{matrix};\frac{1}{2}(t-\tau)s\right]d\tau\right\}dt$$

$$=\frac{\Gamma(d)\Gamma(d')}{s^{d+d'}}\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\frac{\sqrt{\pi}\Gamma\left(\frac{1}{2}(a'+b'+1)\right)}{\Gamma\left(\frac{1}{2}(a'+1)\right)\Gamma\left(\frac{1}{2}(b'+1)\right)},$$

$$\int_0^\infty e^{-st}\left\{\int_0^t \tau^{d-1}{}_3F_3\left[\begin{matrix}a,b,1\\c,d,2\end{matrix};\tau s\right](t-\tau)^{d'-1}{}_3F_3\left[\begin{matrix}a',b',1\\(a'+b'+1)/2,d',2\end{matrix};\frac{1}{2}(t-\tau)s\right]d\tau\right\}dt$$

$$=\frac{\Gamma(d)\Gamma(d')}{s^{d+d'}}\frac{(c-1)}{(a-1)(b-1)}\frac{(a'+b'-1)}{(a'-1)(b'-1)}$$

$$\times\left\{\frac{\Gamma(c-1)\Gamma(c-a-b+1)}{\Gamma(c-a)\Gamma(c-b)}-1\right\}\left\{\frac{\sqrt{\pi}\Gamma\left(\frac{1}{2}(a'+'b-1)\right)}{\Gamma\left(\frac{a'}{2}\right)\Gamma\left(\frac{b'}{2}\right)}-1\right\},$$

and

$$\int_0^\infty e^{-st}\left\{\int_0^t \tau^{d-1}{}_3F_3\left[\begin{matrix}a,b,1\\c,d,3\end{matrix};\tau s\right](t-\tau)^{d'-1}{}_3F_3\left[\begin{matrix}a',b',1\\(a'+b'+1)/2,d',3\end{matrix};\frac{1}{2}(t-\tau)s\right]d\tau\right\}dt$$

$$=\frac{\Gamma(d)\Gamma(d')}{s^{d+d'}}\frac{2(c-2)_2}{(a-2)_2(b-2)_2}\frac{2(a'-b'+1)(a'+b'-3)}{(a'-2)_2(b'-2)_2}$$

$$\times\left\{\frac{\Gamma(c-2)\Gamma(c-a-b+2)}{\Gamma(c-a)\Gamma(c-b)}-\frac{ab+c-2a-2b+2}{c-2}\right\}$$

$$\times\left\{\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}(a'+b'-3)\right)}{\Gamma\left(\frac{1}{2}(a'-1)\right)\Gamma\left(\frac{1}{2}(b'-1)\right)}-\frac{a'b'-a'-b'+1}{a'+b'-3}\right\}.$$

Theorem 3.4: For $\Re(d)>0$, $\Re(d')>0$ and $\Re(s)>0$, following result holds true:

$$\int_0^\infty e^{-st}\left\{\int_0^t \tau^{d-1}{}_3F_3\left[\begin{matrix}a,b,1\\c,d,m\end{matrix};\tau s\right](t-\tau)^{d'-1}{}_3F_3\left[\begin{matrix}a',2m-a'-1,1\\b',d',m\end{matrix};\frac{1}{2}(t-\tau)s\right]d\tau\right\}dt \quad (3.4)$$

$$=\frac{\Gamma(d)\Gamma(d')}{s^{d+d'}}\Omega_1(a,b,c)\,\Omega_4(a',2m-a'-1,b').$$

Where Ω_1 and Ω_4 is given by Eq. (2.2) and Eq. (2.5) respectively.

Corollary 4. If we take $m = 1,2,3$ in Theorem 3.4, we respectively get the following new integrals:

$$\int_0^\infty e^{-st}\left\{\int_0^t \tau^{d-1}\,{}_2F_2\left[\begin{matrix}a,b\\ c,d\end{matrix};\tau s\right](t-\tau)^{d'-1}\,{}_2F_2\left[\begin{matrix}a',1-a'\\ b',d'\end{matrix};\frac{1}{2}(t-\tau)s\right]d\tau\right\}dt$$

$$=\frac{\Gamma(d)\Gamma(d')}{s^{d+d'}}\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\frac{\Gamma\left(\frac{1}{2}b'\right)\Gamma\left(\frac{1}{2}(b'+1)\right)}{\Gamma\left(\frac{1}{2}(a'+b')\right)\Gamma\left(\frac{1}{2}(b'-a'+1)\right)},$$

$$\int_0^\infty e^{-st}\left\{\int_0^t \tau^{d-1}\,{}_3F_3\left[\begin{matrix}a,b,1\\ c,d,2\end{matrix};\tau s\right](t-\tau)^{d'-1}\,{}_3F_3\left[\begin{matrix}a',3-a',1\\ b',d',2\end{matrix};\frac{1}{2}(t-\tau)s\right]d\tau\right\}dt$$

$$=\frac{\Gamma(d)\Gamma(d')}{s^{d+d'}}\frac{(c-1)}{(a-1)(b-1)}\left\{\frac{\Gamma(c-1)\Gamma(c-a-b+1)}{\Gamma(c-a)\Gamma(c-b)}-1\right\}$$

$$\times\frac{2(1-b')}{(1-a')_2}\left\{\frac{\Gamma\left(\frac{1}{2}(b'-1)\right)\Gamma\left(\frac{b'}{2}\right)}{\Gamma\left(\frac{1}{2}(a'+b')-1\right)\Gamma\left(\frac{1}{2}(b'-a'+1)\right)}-1\right\},$$

and

$$\int_0^\infty e^{-st}\left\{\int_0^t \tau^{d-1}\,{}_3F_3\left[\begin{matrix}a,b,1\\ c,d,3\end{matrix};\tau s\right](t-\tau)^{d'-1}\,{}_3F_3\left[\begin{matrix}a',5-a',1\\ b',d',3\end{matrix};\frac{1}{2}(t-\tau)s\right]d\tau\right\}dt$$

$$=\frac{\Gamma(d)\Gamma(d')}{s^{d+d'}}\frac{16(c-2)_2}{(a-2)_2(b-2)_2}\frac{(b'-2)_2}{(a'-4)_4}$$

$$\times\left\{\frac{\Gamma(c-2)\Gamma(c-a-b+2)}{\Gamma(c-a)\Gamma(c-b)}-\frac{ab+c-2a-2b+2}{c-2}\right\}$$

$$\times\left\{\frac{\Gamma\left(\frac{1}{2}(b'-1)\right)\Gamma\left(\frac{1}{2}(b'-2)\right)}{\Gamma\left(\frac{1}{2}(a'+b')-2\right)\Gamma\left(\frac{1}{2}(b'-a'+1)\right)}-\frac{5a'-(a')^2+2b'-10}{2(b'-2)}\right\}.$$

Theorem 3.5: For $\Re(d) > 0$, $\Re(d') > 0$ and $\Re(s) > 0$, following result holds true:

$$\int_0^\infty e^{-st}\left\{\begin{array}{l}\int_0^t \tau^{d-1}\,{}_3F_3\left[\begin{array}{c}a,b,1\\ a-b+m,d,m\end{array};\ -\tau s\right](t-\tau)^{d'-1}\\ \times{}_3F_3\left[\begin{array}{c}a',b',1\\ a'-b'+m,d',m\end{array};\ -(t-\tau)s\right]d\tau\end{array}\right\}dt=\frac{\Gamma(d)\Gamma(d')}{s^{d+d'}} \tag{3.5}$$

$$\times\Omega_2(a,b,a-b+m)\,\Omega_2(a',b',a'-b'+m).$$

Where Ω_2 is given by Eq. (2.3).

Corollary 5. If we take $m = 1,2,3$ in Theorem 3.5, we respectively get the following new integrals:

$$\int_0^\infty e^{-st}\left\{\begin{array}{l}\int_0^t \tau^{d-1}\,{}_2F_2\left[\begin{array}{c}a,b\\ 1+a-b,d\end{array};\ -\tau s\right](t-\tau)^{d'-1}\\ \times{}_2F_2\left[\begin{array}{c}a',b'\\ 1+a'-b',d'\end{array};\ -(t-\tau)s\right]d\tau\end{array}\right\}dt$$

$$=\frac{\Gamma(d)\Gamma(d')}{s^{d+d'}}\frac{\Gamma(1+a-b)\Gamma\left(1+\frac{a}{2}\right)}{\Gamma\left(1-b+\frac{a}{2}\right)\Gamma(1+a)}\frac{\Gamma(1+a'-b')\Gamma\left(1+\frac{a'}{2}\right)}{\Gamma\left(1-b'+\frac{a'}{2}\right)\Gamma(1+a')},$$

$$\int_0^\infty e^{-st}\left\{\begin{array}{l}\int_0^t \tau^{d-1}\,{}_3F_3\left[\begin{array}{c}a,b,1\\ a-b+2,d,2\end{array};\ -\tau s\right](t-\tau)^{d'-1}\\ \times{}_3F_3\left[\begin{array}{c}a',b',1\\ a'-b'+2,d',2\end{array};\ -(t-\tau)s\right]d\tau\end{array}\right\}dt$$

$$=\frac{\Gamma(d)\Gamma(d')}{s^{d+d'}}\ \frac{(a-b+1)}{(a-1)(b-1)}\frac{(a'-b'+1)}{(a'-1)(b'-1)}$$

$$\times\left\{1-\frac{\Gamma(1+a-b)\Gamma\left(\frac{a}{2}+\frac{1}{2}\right)}{\Gamma(a)\Gamma\left(\frac{a}{2}-b+\frac{3}{2}\right)}\right\}\left\{1-\frac{\Gamma(1+a'-b')\Gamma\left(\frac{a'}{2}+\frac{1}{2}\right)}{\Gamma(a')\Gamma\left(\frac{a'}{2}-b'+\frac{3}{2}\right)}\right\},$$

and

$$\int_0^\infty e^{-st}\left\{\begin{array}{l}\int_0^t \tau^{d-1}\,{}_3F_3\left[\begin{array}{c}a,b,1\\ a-b+3,d,3\end{array};-\tau s\right](t-\tau)^{d'-1}\\ \times{}_3F_3\left[\begin{array}{c}a',b',1\\ a'-b'+3,d',3\end{array};-(t-\tau)s\right]d\tau\end{array}\right\}dt$$

$$=\frac{\Gamma(d)\Gamma(d')}{s^{d+d'}}\frac{4(a-b+1)_2}{(a-2)_2(b-2)_2}\left\{\frac{\Gamma(a-b+1)\Gamma\left(\frac{a}{2}\right)}{\Gamma(a-1)\Gamma\left(-b+2+\frac{a}{2}\right)}-\frac{3a-b-ab-3}{a-b+1}\right\}$$

$$\times\frac{(a'-b'+1)_2}{(a'-2)_2(b'-2)_2}\left\{\frac{\Gamma(a'-b'+1)\Gamma\left(\frac{a'}{2}\right)}{\Gamma(a'-1)\Gamma\left(-b'+2+\frac{a'}{2}\right)}-\frac{3a'-b'-a'b'-3}{a'-b'+1}\right\}.$$

Theorem 3.6: For $\Re(d)>0$, $\Re(d')>0$ and $\Re(s)>0$, following result holds true:

$$\int_0^\infty e^{-st}\left\{\begin{array}{l}\int_0^t \tau^{d-1}\,{}_3F_3\left[\begin{array}{c}a,b,1\\ a-b+m,d,m\end{array};-\tau s\right](t-\tau)^{d'-1}\\ {}_3F_3\left[\begin{array}{c}a',b',1\\ (a'-b'+1)/2,d',m\end{array};\frac{1}{2}(t-\tau)s\right]d\tau\end{array}\right\}dt \tag{3.6}$$

$$=\frac{\Gamma(d)\Gamma(d')}{s^{d+d'}}\Omega_2(a,b,a-b+m)\,\Omega_3\left(a',b',(a'-b'+1)/2,m\right).$$

Where Ω_2 and Ω_3 is given by Eq. (2.3) and Eq. (2.4), respectively.

Corollary 6. If we take $m=1,2,3$ in Theorem 3.6, we respectively get the following new integrals:

$$\int_0^\infty e^{-st}\left\{\begin{array}{l}\int_0^t \tau^{d-1}\,{}_2F_2\left[\begin{array}{c}a,b\\ a-b+1,d\end{array};-\tau s\right](t-\tau)^{d'-1}\\ {}_2F_2\left[\begin{array}{c}a',b'\\ (a'-b'+1)/2,d'\end{array};\frac{1}{2}(t-\tau)s\right]d\tau\end{array}\right\}dt$$

$$=\frac{\Gamma(d)\Gamma(d')}{s^{d+d'}}\frac{\Gamma(1+a-b)\Gamma\left(1+\frac{a}{2}\right)}{\Gamma\left(1-b+\frac{a}{2}\right)\Gamma(1+a)}\frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}(a'+b'+1)\right)}{\Gamma\left(\frac{1}{2}(a'+1)\right)\Gamma\left(\frac{1}{2}(b'+1)\right)},$$

$$\int_0^\infty e^{-st}\left\{\begin{array}{l}\int_0^t \tau^{d-1}\,{}_3F_3\left[\begin{array}{c}a,b,1\\ a-b+1,2,d\end{array};-\tau s\right](t-\tau)^{d'-1}\\ {}_3F_3\left[\begin{array}{c}a',b',1\\ (a'-b'+1)/2,2,d'\end{array};\frac{1}{2}(t-\tau)s\right]d\tau\end{array}\right\}dt$$

$$=\frac{\Gamma(d)\Gamma(d')}{s^{d+d'}}\ \frac{(a-b+1)}{(a-1)(b-1)}\frac{(a'+b'-1)}{(a'-1)(b'-1)}$$

$$\times\left\{1-\frac{\Gamma(1+a-b)\Gamma\left(\frac{a}{2}+\frac{1}{2}\right)}{\Gamma(a)\Gamma\left(\frac{a}{2}-b+\frac{3}{2}\right)}\right\}\left\{\frac{\sqrt{\pi}\Gamma\left(\frac{1}{2}(a'+'b-1)\right)}{\Gamma\left(\frac{a'}{2}\right)\Gamma\left(\frac{b'}{2}\right)}-1\right\},$$

and

$$\int_0^\infty e^{-st}\left\{\begin{array}{l}\int_0^t \tau^{d-1}\,{}_3F_3\left[\begin{array}{c}a,b,1\\ a-b+1,3,d\end{array};-\tau s\right](t-\tau)^{d'-1}\\ \times{}_3F_3\left[\begin{array}{c}a',b',1\\ (a'-b'+1)/2,3,d'\end{array};\frac{1}{2}(t-\tau)s\right]d\tau\end{array}\right\}dt=\frac{\Gamma(d)\Gamma(d')}{s^{d+d'}}$$

$$\times\frac{4(a-b+1)_2}{(a-2)_2(b-2)_2}\frac{(a'-b'+1)(a'+b'-3)}{(a'-2)_2(b'-2)_2}$$

$$\times\left\{\frac{\Gamma(a-b+1)\Gamma\left(\frac{a}{2}\right)}{\Gamma(a-1)\Gamma\left(-b+2+\frac{a}{2}\right)}-\frac{3a-b-ab-3}{a-b+1}\right\}$$

$$\times\left\{\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}(a'+b'-3)\right)}{\Gamma\left(\frac{1}{2}(a'-1)\right)\Gamma\left(\frac{1}{2}(b'-1)\right)}-\frac{a'b'-a'-b'+1}{a'+b'-3}\right\}.$$

Theorem .37: For $\Re(d)>0$, $\Re(d')>0$ and $\Re(s)>0$, following result holds true:

$$\int_0^\infty e^{-st}\left\{\begin{array}{l}\int_0^t \tau^{d-1}\,{}_3F_3\left[\begin{array}{c}a,b,1\\ a-b+m,d,m\end{array};-\tau s\right](t-\tau)^{d'-1}\\ \times{}_3F_3\left[\begin{array}{c}a',2m-a'-1,1\\ b',d',m\end{array};\frac{1}{2}(t-\tau)s\right]d\tau\end{array}\right\}dt \tag{3.7}$$

$$=\frac{\Gamma(d)\Gamma(d')}{s^{d+d'}}\Omega_2(a,b,a-b+m)\,\Omega_4\left(a',2m-a'-1,b',m\right).$$

Where Ω_2 and Ω_4 is given by Eq. (2.3) and Eq. (2.5) respectively.

Corollary 7. If we take $m = 1,2,3$ in Theorem 3.7, we respectively get the following new integrals:

$$\int_0^\infty e^{-st}\left\{\begin{array}{l}\int_0^t \tau^{d-1}\,{}_2F_2\left[\begin{array}{c}a,b\\ a-b+1,d\end{array};-\tau s\right](t-\tau)^{d'-1}\\ \times{}_2F_2\left[\begin{array}{c}a',1-a'\\ b',d'\end{array};\frac{1}{2}(t-\tau)s\right]\end{array}\right\}dt$$

$$=\frac{\Gamma(d)\Gamma(d')}{s^{d+d'}}\frac{\Gamma(1+a-b)\Gamma\left(1+\frac{a}{2}\right)}{\Gamma\left(1-b+\frac{a}{2}\right)\Gamma(1+a)}\frac{\Gamma\left(\frac{1}{2}b'\right)\Gamma\left(\frac{1}{2}(b'+1)\right)}{\Gamma\left(\frac{1}{2}(a'+b')\right)\Gamma\left(\frac{1}{2}(b'-a'+1)\right)},$$

$$\int_0^\infty e^{-st}\left\{\begin{array}{l}\int_0^t \tau^{d-1}\,{}_3F_3\left[\begin{array}{c}a,b,1\\ a-b+1,2,d\end{array};-\tau s\right](t-\tau)^{d'-1}\\ {}_3F_3\left[\begin{array}{c}a',3-a',1\\ b',d',2\end{array};\frac{1}{2}(t-\tau)s\right]d\tau\end{array}\right\}dt$$

$$=\frac{\Gamma(d)\Gamma(d')}{s^{d+d'}}\;\frac{(a-b+1)}{(a-1)(b-1)}\frac{2(1-b')}{(1-a')_2}$$

and

$$\times\left\{1-\frac{\Gamma(1+a-b)\Gamma\left(\frac{a}{2}+\frac{1}{2}\right)}{\Gamma(a)\Gamma\left(\frac{a}{2}-b+\frac{3}{2}\right)}\right\}\left\{\frac{\Gamma\left(\frac{1}{2}(b'-1)\right)\Gamma\left(\frac{b'}{2}\right)}{\Gamma\left(\frac{1}{2}(a'+b')-1\right)\Gamma\left(\frac{1}{2}(b'-a'+1)\right)}-1\right\},$$

$$\int_0^\infty e^{-st}\left\{\begin{array}{l}\int_0^t \tau^{d-1}\,{}_3F_3\left[\begin{array}{c}a,b,1\\ a-b+1,3,d\end{array};-\tau s\right](t-\tau)^{d'-1}\\ \times{}_3F_3\left[\begin{array}{c}a',5-a',1\\ b',d',3\end{array};\frac{1}{2}(t-\tau)s\right]d\tau\end{array}\right\}dt=\frac{\Gamma(d)\Gamma(d')}{s^{d+d'}}$$

$$\times\frac{16(a-b+1)_2}{(a-2)_2(b-2)_2}\frac{(b'-2)_2}{(a'-4)_4}\left\{\frac{\Gamma(a-b+1)\Gamma\left(\frac{a}{2}\right)}{\Gamma(a-1)\Gamma\left(-b+2+\frac{a}{2}\right)}-\frac{3a-b-ab-3}{a-b+1}\right\}$$

$$\times\left\{\frac{\Gamma\left(\frac{1}{2}(b'-1)\right)\Gamma\left(\frac{1}{2}(b'-2)\right)}{\Gamma\left(\frac{1}{2}(a'+b')-2\right)\Gamma\left(\frac{1}{2}(b'-a'+1)\right)}-\frac{5a'-(a')^2+2b'-10}{2(b'-2)}\right\}.$$

Theorem 3.8: For $\Re(d) > 0$, $\Re(d') > 0$ and $\Re(s) > 0$, following result holds true:

$$\int_0^\infty e^{-st}\left\{\begin{array}{l}\int_0^t \tau^{d-1}\,{}_3F_3\left[\begin{array}{c}a,b,1\\(a+b+1)/2,d,m\end{array};\frac{1}{2}\tau s\right](t-\tau)^{d'-1}\\ \times{}_3F_3\left[\begin{array}{c}a',b',1\\(a'+b'+1)/2,d',m\end{array};\frac{1}{2}(t-\tau)s\right]d\tau\end{array}\right\}dt \tag{3.8}$$

$$=\frac{\Gamma(d)\Gamma(d')}{s^{d+d'}}\Omega_3(a,b,(a-b+1)/2,m)\,\Omega_3\left(a',b',(a'-b'+1)/2,m\right).$$

Where Ω_3 is given by (2.4).

Corollary 8. If we take $m = 1,2,3$ in Theorem 3.8, we respectively get the following new integrals:

$$\int_0^\infty e^{-st}\left\{\begin{array}{l}\int_0^t \tau^{d-1}\,{}_2F_2\left[\begin{array}{c}a,b\\(a+b+1)/2,d\end{array};\frac{1}{2}\tau s\right](t-\tau)^{d'-1}\\ \times{}_2F_2\left[\begin{array}{c}a',b'\\(a'+b'+1)/2,d'\end{array};\frac{1}{2}(t-\tau)s\right]d\tau\end{array}\right\}dt$$

$$=\frac{\Gamma(d)\Gamma(d')}{s^{d+d'}}\frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}(a+b+1)\right)}{\Gamma\left(\frac{1}{2}(a+1)\right)\Gamma\left(\frac{1}{2}(b+1)\right)}\frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}(a'+b'+1)\right)}{\Gamma\left(\frac{1}{2}(a'+1)\right)\Gamma\left(\frac{1}{2}(b'+1)\right)},$$

$$\int_0^\infty e^{-st}\left\{\begin{array}{l}\int_0^t \tau^{d-1}\,{}_3F_3\left[\begin{array}{c}a,b,1\\(a+b+1)/2,d,2\end{array};\frac{1}{2}\tau s\right](t-\tau)^{d'-1}\\ \times{}_3F_3\left[\begin{array}{c}a',b',1\\(a'+b'+1)/2,d',2\end{array};\frac{1}{2}(t-\tau)s\right]d\tau\end{array}\right\}dt$$

$$=\frac{\Gamma(d)\Gamma(d')}{s^{d+d'}}\frac{(a+b-1)}{(a-1)(b-1)}\frac{(a'+b'-1)}{(a'-1)(b'-1)}$$

$$\times\left\{\frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}(a+b-1)\right)}{\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{b}{2}\right)}-1\right\}\left\{\frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}(a'+'b-1)\right)}{\Gamma\left(\frac{a'}{2}\right)\Gamma\left(\frac{b'}{2}\right)}-1\right\},$$

and

$$\int_0^\infty e^{-st}\left\{\begin{array}{l}\int_0^t \tau^{d-1}{}_3F_3\left[\begin{array}{c}a,b,1\\(a+b+1)/2,d,3\end{array};\frac{1}{2}\tau s\right](t-\tau)^{d'-1}\\ \times{}_3F_3\left[\begin{array}{c}a',b',1\\(a'+b'+1)/2,d',3\end{array};\frac{1}{2}(t-\tau)s\right]d\tau\end{array}\right\}dt$$

$$=\frac{\Gamma(d)\Gamma(d')}{s^{d+d'}}\frac{4(a+b-1)(a+b-3)}{(a-2)_2(b-2)_2}\frac{(a'+b'-1)(a'+b'-3)}{(a'-2)_2(b'-2)_2}$$

$$\times\left\{\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}(a+b-3)\right)}{\Gamma\left(\frac{1}{2}(a-1)\right)\Gamma\left(\frac{1}{2}(b-1)\right)}-\frac{ab-a-b+1}{a+b-3}\right\}$$

$$\times\left\{\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}(a'+b'-3)\right)}{\Gamma\left(\frac{1}{2}(a'-1)\right)\Gamma\left(\frac{1}{2}(b'-1)\right)}-\frac{a'b'-a'-b'+1}{a'+b'-3}\right\}.$$

Theorem 3.9: For $\Re(d) > 0$, $\Re(d') > 0$ and $\Re(s) > 0$, following result holds true:

$$\int_0^\infty e^{-st}\left\{\begin{array}{l}\int_0^t \tau^{d-1}{}_3F_3\left[\begin{array}{c}a,b,1\\(a+b+1)/2,d,m\end{array};\frac{1}{2}\tau s\right](t-\tau)^{d'-1}\\ \times{}_3F_3\left[\begin{array}{c}a',2m-a'-1,1\\b',d',m\end{array};\frac{1}{2}(t-\tau)s\right]d\tau\end{array}\right\}dt \quad (3.9)$$

$$=\frac{\Gamma(d)\Gamma(d')}{s^{d+d'}}\Omega_3(a,b,(a-b+1)/2,m)\,\Omega_4\left(a',2m-a'-1,b',m\right).$$

Where Ω_3 and Ω_4 is given by Eq. (2.4) and Eq. (2.5) respectively.

Corollary 9. If we take $m = 1,2,3$ in Theorem 3.9, we respectively get the following new integrals:

$$\int_0^\infty e^{-st}\left\{\begin{array}{l}\int_0^t \tau^{d-1}{}_2F_2\left[\begin{array}{c}a,b\\(a+b+1)/2,d\end{array};\frac{1}{2}\tau s\right](t-\tau)^{d'-1}\\ \times{}_2F_2\left[\begin{array}{c}a',1-a'\\b',d'\end{array};\frac{1}{2}(t-\tau)s\right]d\tau\end{array}\right\}dt$$

$$=\frac{\Gamma(d)\Gamma(d')}{s^{d+d'}}\frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}(a+b+1)\right)}{\Gamma\left(\frac{1}{2}(a+1)\right)\Gamma\left(\frac{1}{2}(b+1)\right)}\frac{\Gamma\left(\frac{b'}{2}\right)\Gamma\left(\frac{1}{2}(b'+1)\right)}{\Gamma\left(\frac{1}{2}(a'+b')\right)\Gamma\left(\frac{1}{2}(b'-a'+1)\right)},$$

$$\int_0^\infty e^{-st}\left\{\begin{array}{l}\int_0^t \tau^{d-1}\,{}_2F_2\left[\begin{array}{c}a,b,1\\(a+b+1)/2,d,2\end{array};\frac{1}{2}\tau s\right](t-\tau)^{d'-1}\\ \times{}_2F_2\left[\begin{array}{c}a',1-a',1\\b',d',2\end{array};\frac{1}{2}(t-\tau)s\right]d\tau\end{array}\right\}dt$$

$$=\frac{\Gamma(d)\Gamma(d')}{s^{d+d'}}\frac{(a+b-1)}{(a-1)(b-1)}\frac{2(1-b')}{(1-a')_2}$$ and

$$\times\left\{\frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}(a+b-1)\right)}{\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{b}{2}\right)}-1\right\}\left\{\frac{\Gamma\left(\frac{1}{2}(b'-1)\right)\Gamma\left(\frac{b'}{2}\right)}{\Gamma\left(\frac{1}{2}(a'+b')-1\right)\Gamma\left(\frac{1}{2}(b'-a'+1)\right)}-1\right\},$$

$$\int_0^\infty e^{-st}\left\{\begin{array}{l}\int_0^t \tau^{d-1}\,{}_3F_3\left[\begin{array}{c}a,b,1\\(a+b+1)/2,d,3\end{array};\frac{1}{2}\tau s\right](t-\tau)^{d'-1}\\ \times{}_3F_3\left[\begin{array}{c}a',5-a',1\\b',d',3\end{array};\frac{1}{2}(t-\tau)s\right]d\tau\end{array}\right\}dt$$

$$=\frac{\Gamma(d)\Gamma(d')}{s^{d+d'}}\frac{16(a+b-1)(a+b-3)}{(a-2)_2(b-2)_2}\frac{(b'-2)_2}{(a'-4)_2}$$

$$\times\left\{\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}(a+b-3)\right)}{\Gamma\left(\frac{1}{2}(a-1)\right)\Gamma\left(\frac{1}{2}(b-1)\right)}-\frac{ab-a-b+1}{a+b-3}\right\}$$

$$\times\left\{\frac{\Gamma\left(\frac{1}{2}(b'-1)\right)\Gamma\left(\frac{1}{2}(b'-2)\right)}{\Gamma\left(\frac{1}{2}(a'+b')-2\right)\Gamma\left(\frac{1}{2}(b'-a'+1)\right)}-\frac{5a'-(a')^2+2b'-10}{2(b'-2)}\right\}.$$

Theorem 3.10: For $\Re(d) > 0$, $\Re(d') > 0$ and $\Re(s) > 0$, following result holds true:

$$\int_0^\infty e^{-st}\left\{\begin{array}{l}\int_0^t \tau^{d-1}\,{}_3F_3\left[\begin{array}{c}a,2m-a-1,1\\b,d,m\end{array};\frac{1}{2}\tau s\right](t-\tau)^{d'-1}\\ \times{}_3F_3\left[\begin{array}{c}a',2m-a'-1,1\\b',d',m\end{array};\frac{1}{2}(t-\tau)s\right]d\tau\end{array}\right\}dt \tag{3.10}$$

$$=\frac{\Gamma(d)\Gamma(d')}{s^{d+d'}}\Omega_4(a,2m-a-1,b,m)\,\Omega_4\left(a',2m-a'-1,b',m\right).$$

Where Ω_4 is given by Eq. (2.5).

Corollary 10. If we take $m = 1,2,3$ in Theorem 3.10, we respectively get the following new integrals

$$\int_0^\infty e^{-st}\left\{\begin{array}{l}\int_0^t \tau^{d-1}\,{}_2F_2\left[\begin{array}{c}a,1-a\\ b,d\end{array};\ \frac{1}{2}\tau s\right](t-\tau)^{d'-1}\\ \times{}_2F_2\left[\begin{array}{c}a',1-a'\\ b',d'\end{array};\ \frac{1}{2}(t-\tau)s\right]d\tau\end{array}\right\}dt$$

$$=\frac{\Gamma(d)\Gamma(d')}{s^{d+d'}}\frac{\Gamma\left(\frac{b}{2}\right)\Gamma\left(\frac{1}{2}(b+1)\right)}{\Gamma\left(\frac{1}{2}(a+b)\right)\Gamma\left(\frac{1}{2}(b-a+1)\right)}\frac{\Gamma\left(\frac{b'}{2}\right)\Gamma\left(\frac{1}{2}(b'+1)\right)}{\Gamma\left(\frac{1}{2}(a'+b')\right)\Gamma\left(\frac{1}{2}(b'-a'+1)\right)},$$

$$\int_0^\infty e^{-st}\left\{\begin{array}{l}\int_0^t \tau^{d-1}\,{}_3F_3\left[\begin{array}{c}a,3-a,1\\ b,d,2\end{array};\ \frac{1}{2}\tau s\right](t-\tau)^{d'-1}\\ \times{}_3F_3\left[\begin{array}{c}a',3-a',1\\ b',d',2\end{array};\ \frac{1}{2}(t-\tau)s\right]d\tau\end{array}\right\}dt$$

$$=\frac{\Gamma(d)\Gamma(d')}{s^{d+d'}}\frac{4(1-b)}{(1-a)_2}\frac{(1-b')}{(1-a')_2}$$

$$\times\left\{\frac{\Gamma\left(\frac{1}{2}(b-1)\right)\Gamma\left(\frac{b}{2}\right)}{\Gamma\left(\frac{1}{2}(a+b-1)\right)\Gamma\left(\frac{1}{2}(b-a+1)\right)}-1\right\}\left\{\frac{\Gamma\left(\frac{1}{2}(b'-1)\right)\Gamma\left(\frac{b'}{2}\right)}{\Gamma\left(\frac{1}{2}(a'+b')-1\right)\Gamma\left(\frac{1}{2}(b'-a'+1)\right)}-1\right\},$$

and

$$\int_0^\infty e^{-st}\left\{\begin{array}{l}\int_0^t \tau^{d-1}\,{}_3F_3\left[\begin{array}{c}a,5-a,1\\ b,d,2\end{array};\ \frac{1}{2}\tau s\right](t-\tau)^{d'-1}\\ \times{}_3F_3\left[\begin{array}{c}a',5-a',1\\ b',d',2\end{array};\ \frac{1}{2}(t-\tau)s\right]d\tau\end{array}\right\}dt$$

$$=\frac{\Gamma(d)\Gamma(d')}{s^{d+d'}}\frac{64(b-2)_2}{(a-4)_2}\frac{(b'-2)_2}{(a'-4)_2}$$

$$\times\left\{\frac{\Gamma\left(\frac{1}{2}(b-1)\right)\Gamma\left(\frac{1}{2}(b-2)\right)}{\Gamma\left(\frac{1}{2}(a+b)-2\right)\Gamma\left(\frac{1}{2}(b-a+1)\right)}-\frac{5a-a^2+2b-10}{2(b-2)}\right\}$$

$$\times\left\{\frac{\Gamma\left(\frac{1}{2}(b^{'}-1)\right)\Gamma\left(\frac{1}{2}(b^{'}-2)\right)}{\Gamma\left(\frac{1}{2}(a^{'}+b^{'})-2\right)\Gamma\left(\frac{1}{2}(b^{'}-a^{'}+1)\right)}-\frac{5a^{'}-(a^{'})^{2}+2b^{'}-10}{2(b^{'}-2)}\right\}.$$

Remark: for $m = 1$, Theorems 3.1 to 3.10 give the result earlier obtained by Rathie et al. (2021).

4. Conclusion

Utilizing the generalization of the classical summation theorem for the series ${}_2F_1$ established by Masjed-Jamei and Koepf (2018), we present novel and insightful conclusions regarding the Laplace transform of convolution-type integrals with generalized hypergeometric functions. Notably, our results specialize to the established findings of Rathie et al. (2021) for the particular case of m, demonstrating the generality and applicability of our framework. The resulting expressions, characterized by their inherent simplicity and straightforward proofs, offer promising avenues for advancement in diverse fields within mathematical physics and classical analysis.

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