Fixed Point theorem in cone Banach space

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Abstract: In the past few several researchers have come up fixed point thearem in cone banach space. In the paper in are going to delibevate about the concept of one Banach space many previous coupled fixed point theorem are extended and generalized by it. In this study are discussed some outcomes of fixed point cone banach space.

Keywords : Banach Space, coupled fixed point, cone metric space.

Introduction: The concept of come Banach space was presented as a generalization of metric spaces. I 200 Erdal karnapinar introduce cone Banach space he proves and extended some important result of common fixed-point theorem for self-mapping and many other authors are proved some results in cone Banach space.

Definition 6.1.1: Let $F$and $G$are two self-mappings They said to be commuting if

$$FGx=GFx, \qquad \text{for all} x\in X.$$

Definition6.1.2: Let$(X,\|\cdot\|)$be Cone-Normed-Space,Mappings F,G:X⟶X

"Weak-compatible "if they commute at "coincidence points",

$$\text{i.e.} \quad Fx=Gx \quad \Rightarrow FGx=GFx$$

Definition6.1.3: TwoSelfmappings$F$and$G$ofaConenormedspace (X,|| ·||)aresaidtobecompatibleif

$$\lim_{n\to\infty}\|FGx_n - GFx_n\| = 0 \qquad \text{for all } x \in X,$$

where$\{x_n\}$isasequenceinXsuchthat if

$$\lim_{n\to\infty} Fx_n = \lim_{n\to\infty} Gx_n = x \qquad \text{for all } x \in X$$

Definition6.1.4: Let,$F$and$G$betwoselfmappingsonaset X,if

$$Fx = Gx \text{ for some } x \in X$$

then$x$is calledcoincidencepointof $F$and$G$.

Definition6.1.5: Let(X,||·||)isaconeBanachSpace,twomappings $F$and$G$onconeBanachSpaceissatisfytheproperty(E.A.)fora sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} Fx_n = \lim_{n\to\infty} Gx_n = t$$

Definition 6.1.6: Let $(X, \|\cdot\|)$ is a cone Banach Space, two self-mappings F and G on cone Banach Space is satisfy the property (E.A.) for a sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} Fx_n = \lim_{n\to\infty} Gx_n = \psi t$$

For some $t \in X$. Therefore F and G are satisfy the (CLR_G) property.

6.1 Main Result:

Fixed-Point Theorem by Unsing E.A. Property

Theorem 6.2.1: Four self-mapping F, G, H and L defined on Cone Banach Space $(X, \|.\|)$ with $\|x\| = d(x, 0)$ satisfying the condition

$$\|Hx - Ly\| \leq \frac{a}{2} max\{\|Gy - Fx\|, \|Hx - Ly\|, \|Lx - Gy\|\} + \frac{b}{2} max\{\|Fx - Ly\|, \|Ly - Hx\|\} \quad \ldots.(6.1)$$

For all $x, y \in X$; $\quad (1 - \frac{a}{2} - \frac{b}{2}) \leq [0,1)$

(i) $H(X)\subseteq G(X)$and$L(X)\subseteq F(X)$

(ii) $(H,F)$and(L,G)areweaklycompatible.

(iii) Property(E.A.)satisfiedby satisfied by$(H,F)$and (L,G)

ThenF,G,H and L have aunique common fixed point.

Proof: Suppose that property (E.A.)satisfies by the pair (L,G) then there exist a sequence $\{x_n\}$ in Xsuch that

$$\lim_{n\to\infty} L\{x_n\}=\lim_{n\to\infty} G\{x_n\}=t \qquad \text{forsome} t\in X$$

From condition(i) we have$L(X)\subseteq F(X)$ then there existasequence $\{v_n\}$ in Xsuchthat

$$L\{x_n\}=F\{y_n\} \quad \text{hence} \lim_{n\to\infty} F\{y_n\}=t$$

Now we claim that $\lim_{n\to\infty} H\{y_n\}=t$ on the contradiction let usPut $x=$ $y_n$and

$y=x_n$in equation (6.1)

$$\|Hy_n-Lx_n\| \leq \frac{a}{2} max\{\|Gx_n-Fy_n\|,\|Hy_n-Lx_n\|,\|Ly_n-Gx_n\|\} + \frac{b}{2} max\{\|Fy_n-Lx_n\|,\|Lx_n-Hy_n\|\}$$

$$\|Hy_n-Lx_n\| \leq \frac{a}{2} max\{\|Gx_n-Ly_n\|,\|Hy_n-Lx_n\|,\|Ly_n-Gx_n\|\} + \frac{b}{2} max\{\|Ly_n-Lx_n\|,\|Lx_n-Hy_n\|\}$$

We claim that $n\to\infty$

$$\|Hy_n - t\| \leq \frac{a}{2} max\{\|t-t\|, \|Hy_n - t\|, \|t-t\|\}$$
$$+ \frac{b}{2} max\{\|t-t\|, \|t - Hy_n\|\}$$

$$(1 - \frac{a}{2} - \frac{b}{2})\|Hy_n - t\| \leq 0$$

$$H(y_n) = t$$

Hence $\lim_{n\to\infty} H\{y_n\} = \lim_{n\to\infty} F\{y_n\} = t$

Now we assume that $F(X)$ is complete subspace of X and $t = F(w)$ for some

$w \in X$, then

$$\lim_{n\to\infty} L\{x_n\} = \lim_{n\to\infty} G\{x_n\} = \lim_{n\to\infty} H\{x_n\} = \lim_{n\to\infty} F\{y_n\} = t = F(w)$$

We claim that $H(w) = F(w)$, if it is not then we put $x = w$ and $y = x_n$ in equation (6.1).

$$\|Hw - Lx_n\|$$
$$\leq \frac{a}{2} max\{\|Gx_n - Lw\|, \|Hw - Lx_n\|, \|Lx_n - Gx_n\|\}$$
$$+ \frac{b}{2} max\{\|Lw - Lx_n\|, \|Lx_n - Hw\|\}$$

Taking Limit $n \to \infty$, we get

$$\|Hw - t\| \leq \frac{a}{2} max\{\|t - t\|, \|Hw - t\|, \|t - t\|\}$$
$$+ \frac{b}{2} max\{\|t - t\|, \|t - Hw\|\}$$

$$(1 - \frac{a}{2} - \frac{b}{2})\|Hw - t\| = 0$$

$$H(w)=t$$

$$F(w)=H(w)=t$$

Hence w is coincidencepointof (H,F).

Nowfromtheweakcomputability of (F,H) wehave

$HF(w)=FH(w)$ or $Ht=Ft$.

Since $H(X) \subseteq G(X)$ there is an element $z \in X$. such that $H(w) = G(z)$.

Thus $H(w) = F(w) = G(z) = t$

We showthatziscoincidencepointof (L,G) isz,thatis $G(z)=$
$L(z)=t$

Ifnotthenweput $x=w$ and $y=z$ inequation (6.1)

$$\|Hw-Lz\| \leq \frac{a}{2} max\{\|Gz - Lw\|,\|Hw - Lz\|,\|Lz - Gz\|\} + \frac{b}{2} max\{\|Lw - Lz\|,\|Lz - Hw\|\}$$

TakingLimit $n\rightarrow\infty$,weget

$$\|t-Lz\| \leq \frac{a}{2}\{\|t - t\|,\|t - L(z)\|,\|t - Lz\|\} + \frac{b}{2} max\{\|t - Lz\|,\|t\ (z) - Hw\|\}$$

$$\|t-Lz\| \leq \frac{a}{2}\{\|t-L(z)\|\} + \frac{b}{2}\|t-Lz\|$$

$$(1-\frac{a}{2}-\frac{b}{2})\|t - L(z)\| \leq 0$$

$$t=L(z)$$

Clearly $L(z)=G(z)=t$,

z is a coincidence pointof (L,G). Sincethepair (L,G) are weak compatible

$$\Longrightarrow GL(w)=LG(w) or Lt=Gt$$

Hence F,G,H and L have acommoncoincidencepoint t.

Next,we prove that common fixed point of F,G,H and L.So,weput that $x= w$ and $y = t$ in equation (6.1)

$$\|Hw-Lt\| \leq \frac{a}{2} max\{\|Gt-Fw\|,\|Hw-Lt\|,\|Lt-Gt\|\} + \frac{b}{2} max\{\|Lw-Lt\|,\|Lt-Hw\|\}$$

$$\|t-L(t)\| \leq \frac{a}{2} max\{\|Lt-t\|,\|t-Lt\|,\|t-t\|\} + \frac{b}{2} max\{\|t-Lt\|,\|t-Lt\|\}$$

$$\|t-Lt\| \leq 0 \Longrightarrow t=L(t)$$

Clearly $F(t)=H(t)=L(t)=G(t)=t$

Hence t iscommonfixedpointof F,H,G and L.

Uniqueness

Let t' beanotherfixed point.Weput $x=t'$ and $y=t$ in(6.1)

$$\|Ht'-Lt\| \leq \frac{a}{2} max\{\|Gt-Ft'\|,\|Ht'-Lt\|,\|Lt'-Gt\|\} + \frac{b}{2} max\{\|Ft'-Lt\|, \|Lt-Ht'\|\}$$

$$\|t' - t\| \leq \frac{a}{2} max\{\|t - t'\|, \|t' - t\|, \|t' - t\|\} + \frac{b}{2} max\{\|t' - t\|, \|t - t'\|\}$$

$$\|t' - t\| \leq \frac{a}{2}\|t' - t\| + \frac{b}{2}\|t' - t\|$$

$$(1 - \frac{a}{2} - \frac{b}{2})\|t' - t\| \leq 0$$

Since $(1 - \frac{a}{2} - \frac{b}{2}) \neq 0$

$$t' = t$$

Hence t is a unique common fixed point.

We assume that $G(X)$ is a complete subspace of X, a similar argument obtains. If the pair (H, F) satisfies property (E.A.) then we get similar result.

Fixed Point Theorem by Using CLR Property:

Theorem 6.2.2: Two self-mappings F, G, H and L be defined on Cone Banach Space $(X, \|.\|)$ with $\|x\| = d(x, 0)$ satisfying the condition

$$\|Hx - Ly\| \leq \frac{a}{2} max\{\|Gy - Fx\|, \|Hx - Ly\|, \|Ly - Gy\|\} + \frac{b}{2} max\{\|Fx - Ly\|, \|Ly - Hx\|\} \quad ....(6.2)$$

Where a and b are non-negative and $(1 - \frac{a}{2} - \frac{b}{2}) < 1$

(i) $H(X) \subseteq G(X)$ and $L(X) \subseteq F(X)$

(ii) The pair (H, F) and (L, G) are weakly compatible.

(iii) Thepair(L,G)or(H,F)satisfiedby$(CLR_L)$and (CLR_H)Property.

Then$F,G,H$and$L$haveauniquecommonfixedpoint.

Proof: First,weassumethatthepair(L,G)satisfied the (CLR_L)Propertythenthereexistthesequence$\{x_n\}$in$X$suchthat

$$\lim_{n\to\infty} Lx_n=\lim_{n\to\infty} Gx_n=Lx \text{ for some } x\in X$$

Further, since $L(X) \subseteq F(X)$ We have$Lx = Fw$for some$w \in X$.

We claim that

$Hw=Fw=t$(say).If notthen $x=w$and $y=x_n$in(6.2)

$$\|Hw-Lx_n\| \leq \frac{a}{2} max\{\|Gx_n-Fw\|,\|Hw-Lx_n\|,\|Lx_n-Gx_n\|\} + \frac{b}{2} max\{\|Fw-Lx_n\|,\|Lx_n-Hw\|\}$$

$$\|Hw-Lx_n\| \leq \frac{a}{2} max\{\|Lx - Lx\|,\|Hw - Lx\|,\|Lx -Lx\| \} + \frac{b}{2} max\{\|Lx -Lx\|,\|Lx -Hw\| \}$$

$$\|Hw-Lx_n\| \leq \frac{a}{2}\|Hw-Lx\| + \frac{b}{2}\|Lx-Hw\|$$

$$\|Hw-Lx_n\|\left(1-\frac{a}{2}-\frac{b}{2}\right)\leq 0$$

$$H(w)=L(x)$$

Hence $Hw=Lx$ implies that $Fw=Hw=Lx=t$

Hence w is coincidence point of H and F.

Since the pair (H,F) is weak compatible

$$\Longrightarrow HFw = Fhw = Ht =$$

Ft Further Since $H(X) \subseteq G(X)$, there exist some $z \in X$ Such that $H(w) = G(z)$

We claim that $L(z)=t$

On the contradiction we put, $x=w$ and $y=z$ in equation (6.2)

$$\|Hw-Lz\| \leq \frac{a}{2} max\{\|Gz - Fw\|, \|Hw - Lz\|, \|Lz - Gz\|\} + \frac{b}{2} max\{\|Fw - Lz\|, \|Lz - Hw\|\}$$

$$\|Hw-Lz\| \leq \frac{a}{2} max\{\|Hw - Hw\|, \|Hw - Lz\|, \|Lz - Hw\|\} + \frac{b}{2} max\{\|Hw - Lz\|, \|Lz - Hw\|\}$$

$$\|Hw-Lz\| \leq \frac{a}{2} \|Hw-Lz\| + b\|Hw-Lz\|$$

$$\left(1-\frac{a}{2}-\frac{b}{2}\right)\|Hw - Lz\| \leq 0$$

$$H(w)=L(z)$$

$$\Longrightarrow t = L(z)$$

Hence $Lz = t$, hence $Fw = Hw = Lz = Gz = t$ It shows that z is coincidences point G.

Also,theweakcompatibilityof(L,G)implies that

$$LGz=Glz=Lt=Gt$$

$x=w$and$y=t$in equation(1).

$$\|Hw-Lt\| \le \frac{a}{2} max\{\|Gw-Fw\|,\|Hw-Lt\|,\|Lt-Gt\|\}$$

$$+\frac{b}{2} max\{\|Fw-Lt\|,\|Lt-Hw\|\}$$

$$\|t-Lt\| \le \frac{a}{2} max\{0,\|t-Lt\|,\|Lt-t\|\}$$

$$+\frac{b}{2} max\{\|t-Lt\|,\|Lt-t\|\}$$

$$(1-\frac{a}{2}-\frac{b}{2})\|t-Lt\| \le 0$$

$$t=Lt$$

Hence$Ft=Ht=Lt=Gt=t$.

Itshowsthattiscommonfixedpointof$F,G,H$and$L$.

Lett'beanotherfixedpointofmappings $F,G,H$and$L$.Letusput $x=t'$and$y=t$in(6.2)

$$\|Ht'-Lt\| \le \frac{a}{2} max\{\|Gt-Ft'\|,\|Ht'-Lt\|,\|Lt-Gt\|\}$$

$$+\frac{b}{2} max\{\|Ft'-Lt\|,\|Lt-Ht'\|\}$$

$$\|t'-t\| \le \frac{a}{2} max\{\|t-t'\|,\|t'-t\|,\|t-t\|\}$$

$$+\frac{b}{2} max\{\|t'-t\|,\|t-t'\|\}$$

$$\|t'-t\| \leq \frac{a}{2}\|t-t'\| + \frac{b}{2}\|t-t'\|$$

$$\left(1-\frac{a}{2}-\frac{b}{2}\right)\|t'-t\| \leq 0$$

Since $\left(1-\frac{a}{2}-\frac{b}{2}\right) \neq 0$

$$t' = t$$

Hencetisauniquecommonfixed point.

Similarly,the argumentthatthepair (H,F) satisfythe (CLR_H) Property will also give the unique common fixed point of F ,G ,$H$and L.

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