FIXED POINTS THEOREMS FOR WEAKLY COMPATIBLE MAPS WITH E.A. AND CLR PROPERTIES

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Abstract:-In this paper, we are going to explore **f**ixed-point theorems about weakly compatible maps possessing the (E.A.) and CLR properties in the context of metric spaces. Weextend and generalize existing results by establishing conditions under which a common fixed point exists for such mappings. By introducing new techniques and utilizing the concepts of weak compatibility, we develop results that unify and refine several known fixed-point theorems.

Keywords:-Cone Banach space, Common Fixed points, Compatible mapping, weakly compatible, property(E.A.), CLR (Common Limit Range) property.

1. INTRODUCTION:-

Huang and Chang [4] gave the notion of cone metric space, replacing the set of real numbers by ordered Banach Space, and introduced some fixed-point theorems for function satisfying contractive conditions in Banach Spaces. Sh. Rezapour and R. Hamalbarani [6] were the results of [4] by omitting the normality condition, which is a milestone in developing fixed point theory in cone metric space. After that several articles on fixed point theorems in cone metric space were obtained by different mathematicians such as M. Abbas , G. Junck [5] , D. Ilic [1] etc. Some results on fixed point theorems have been extended to Cone Banach Space. E. Karapinar [2]. Recently, some common fixed-point theorems have been extended in probabilistic metric spaces and fuzzy metric spaces by the E.A. property under weak compatibility[8, 9]. Sintunavrat and Kumam[10] prove some common fixed-point theorem under weak compatibility. Recently many mathematicians prove some fixed-point theorem in various spaces. [11,12]

2. PRELIMINARIES & DEFINITION

Definition 2.1. Let $(E, \|\cdot\|)$ be a real Banach space. A subset $P \subseteq E$ is said to be a cone if and only if

(i) P is closed, nonempty and $P \neq \{0\}$

(ii) $a, b \epsilon R, a, b \geq 0, \; x, y \epsilon P$ implies $ax, by \epsilon P$

(iii) $P \cap (-P) = \{0\}$

For a given cone **P** subset of E, we define a partial ordering ≤ with respect to **P** by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$ while $x << y$ will stand for $y - x \in int\, P$ where $int\, P$ denotes the interior of **P** and is assumed to be nonempty.

Definition 2.2. [4] Let X be a nonempty set. Suppose that the mapping $d : X \times X \rightarrow E$ satisfies

(i) $0 \leq d(x, y)$ for every $x, y \epsilon X, d(x, y) = 0$ if and only if $x = y$.

(ii) $d(x, y) = d(y, x)$ for every $x, y \epsilon X$.

(iii) $d(x, y) \leq d(x, z) + d(z, y)$ for every $x, y, z \epsilon X$.

Then d is a cone metric on X and *(X, d)* is a cone metric space.

Definition 2.3. [8] Two self-maps A and B on a cone-normed space $(X, \|\cdot\|)$ are said to be weak-compatible if they commute at their coincidence points, i.e. $Ax = Bx$ implies

$ABx = BAx$.

Definition 2.4. [8] Let X be a vector space over R. Suppose the mapping $\|\cdot\| : X \rightarrow E$ satisfies

(i) $\|x\| \geq 0$ for all $x \epsilon X$

(ii) $\|x\| = 0$ if and only if $x = 0$

(iii) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$

(iv) $\|kx\| = |\, k \,|\, \|x\|$ for all $k \in R$.

Then $\|\cdot\|$ is called a norm on X, and $(X, \|\cdot\|)$ is called a cone-normed space. Each cone normed space is a cone metric space with metric defined by $d(x, y) = \|x - y\|$

Definition 2.5. [8] Let $(X, \|\cdot\|)$ be a cone-normed space, $x \in X$ and $\{x_n\}$ a sequence in X. Then

(i). $\{x_n\}$Converges to x if for every $c \in E$ with $0 \ll c$ there is a natural number N such that.

$$\|x_n - x\| \leq c \text{ for all } n \geq N$$

We shall denote it by $\lim_{n\rightarrow\infty} x_n = x$ or $x_n \rightarrow x$.

(ii). $\{x_n\}$ is a Cauchy sequence, if for every $c \epsilon E$ with $0 \ll c$ there is a natural number N such that

$$\|x_n - x_m\| \leq c \text{ for all } n, m \geq N$$

(iii). $(X, \|\cdot\|)$ is a complete cone normed space if every Cauchy sequence is convergent. A complete cone normed space is called a Cone Banach space.

Definition 2.6. [6] Let F and G be self-mappings on a cone-normed space $(X, \|\cdot\|)$, they are said to be compatible if $\lim_{n\to\infty} \|FG(x_n) - GF(x_n)\| = 0$ for every sequence $\{x_n\}$ in X with $\lim_{n\to\infty} F(x_n) = \lim_{n\to\infty} G(x_n) = y$ for some point y in X.

Proposition 2.7. [8] Let $(X, \|\cdot\|)$ be a cone-normed space. P be a normal cone with constant K. Let $\{x_n\}$ be a sequence in X. Then

(i) $\{x_n\}$ converges to x if and only if $\|x_n - x\| \to 0$ as $n \to \infty$.

(ii) $\{x_n\}$ is a Cauchy sequence if and only $\|x_n - x_m\| \to 0$ as $n, m \to \infty$.

(iii) if the $\{x_n\}$ converges to x and $\{y_n\}$ converges to y then $\|x_n - y_n\| \to \|x - y\|$

Proposition 2.8. [6] Let f and g be compatible mappings on a cone-normed space $(X, \|\cdot\|)$ such that $\lim_{n\to\infty} F(x_n) = \lim_{n\to\infty} G(x_n)$ for some point y in X and for every sequence $\{x_n\}$ in X. Then $\lim_{n\to\infty} F(x_n) = F(y)$. if F is continuous.

Theorem 2.10. [6] Let f, g, h, l be mappings on Cone Banach space into $(X, \|\cdot\|)$ itself, with $\|x\| = d(x, 0)$ satisfying the conditions.

(1) $\|hx - ly\| \leq a\|fx - hx\| + b\|fx - ly\| + c\|gy - ly\|$ for all $x, y \in X$,

$$a, b, c \geq 0, a + 2b + c < 1.$$

(2) f and g are onto mapping,

(3) f is continuous ,

(4) f and h; g and l commute

Then f, g, h and l have a unique common fixed point.

Definition 2.11.: Let $(X, \|\cdot\|)$ is a cone Banach Space, two mappings $\emptyset$ and ψ on cone Banach Space is satisfy the property (E.A.) for a sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} \emptyset x_n = \lim_{n\to\infty} \psi x_n = t$$

For some $t \in$ X

Example 2.12: Let $X = [0,1]$, Define $\emptyset, \psi :$ X → X such that

$$\emptyset(x) = 3 - 2x \text{ and } \psi(x) = \frac{4-x}{3}$$

Consider a sequence $x_n = 1 + \frac{1}{n}$ we have,

$$\lim_{n\to\infty} \emptyset(x_n) = 3 - 2 - \frac{2}{n} = 1$$

$$\lim_{n\to\infty} \psi(x_n) = \frac{4}{3} - \frac{1}{3} - \frac{1}{2n} = 1$$

Then$\emptyset$ and ψ satisfies property E.A.

Definition 2.13.: Let(X , $\|\cdot\|$) is a cone Banach Space, two self-mappings $\emptyset$ and ψ on cone Banach Space is satisfy the CLR property for a sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} \emptyset x_n = \lim_{n\to\infty} \psi x_n = \psi \boldsymbol{t}$$

For some $t \in$ X. Therefore $\emptyset$ and ψ are satisfy the (CLR_ψ) property.

Example 2.14: LetX $= [0, \infty)$ be a usual metric space Define $\emptyset, \psi :$ X $\to$ X such that

$$\emptyset(u) = 3u + 1 \text{ and } \psi(u) = 4u$$

Consider a sequence $u_n = 1 + \frac{1}{n}$ we have,

$$\lim_{n\to\infty} \emptyset(x_n) = \lim_{n\to\infty} \psi(x_n) = 4 = 4.1$$

Then$\emptyset$ and ψ satisfies property (CLR_ψ)

3. Main Result:

Theorem3.1: Four self-mapping F, G, H and L defined on Cone Banach Space $(X, \|.\|)$with $\|u\| = d(u, 0)$ satisfying the condition

$$\|Hx - Ly\| \leq \frac{k_1}{2} max\{\|Gy - Fx\|, \|Hx - Ly\|, \|Lx - Gy\|\} + \frac{k_2}{2}\{\|Fx - Ly\|, \|Ly - Hx\|\}$$

(3.1)

For all $u, v \in X$; $\left(1-\frac{k_1}{2}-\frac{k_2}{2}\right) < [\,0\,,1)$

(i) $H(X) \subseteq G(X)$ and $L(X) \subseteq F(X)$

(ii) (H, F) and (L, G) are weakly compatible.

(iii) Property (E.A.) satisfied by satisfied by (H, F) and (L, G)

Then F , G , H and L have a unique common fixed point.

Proof: Suppose that property (E.A.) satisfies by the pair (L, G) then there exist a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} L\{x_n\} = \lim_{n\to\infty} G\{x_n\} = t \quad \text{for some } t \in X$$

From condition (i) we have $L(X) \subseteq F(X)$ then there exist a sequence $\{v_n\}$ in X such that

$$L\{x_n\} = F\{y_n\} \quad \text{hence} \quad \lim_{n\to\infty} F\{y_n\} = t$$

Now we claim that $\lim_{n\to\infty} H\{y_n\} = t$ on the contradiction let us Put $x = y_n$ and $y = x_n$ in equation (3.1)

$$\|Hy_n - Lx_n\| \le \frac{k_1}{2} max\{\|Gx_n - Fy_n\|, \|Hy_n - Lx_n\|, \|Ly_n - Gx_n\|\} + \frac{k_2}{2}\{\|Fy_n - Lx_n\|, \|Lx_n - Hy_n\|\}$$

$$\|Hy_n - Lx_n\| \le \frac{k_1}{2} max\{\|Gx_n - Ly_n\|, \|Hy_n - Lx_n\|, \|Ly_n - Gx_n\|\} + \frac{k_2}{2}\{\|Ly_n - Lx_n\|, \|Lx_n - Hy_n\|\}$$

We claim that $n \to \infty$

$$\|Hy_n - t\| \le \frac{k_1}{2} max\{\|t - t\|, \|Hy_n - t\|, \|t - t\|\} + \frac{k_2}{2}\{\|t - t\|, \|t - Hy_n\|\}$$

$$\left(1-\frac{k_1}{2}-\frac{k_2}{2}\right)\|Hy_n - t\| \le 0$$

$$H(y_n) = t$$

$$\text{Hence } \lim_{n\to\infty} H\{y_n\} = \lim_{n\to\infty} F\{y_n\} = t$$

Now we assume that $F(X)$ is a complete subspace of X and $t = F(w)$ for some

$w \in X$, then

$$\lim_{n\to\infty} L\{x_n\} = \lim_{n\to\infty} G\{x_n\} = \lim_{n\to\infty} H\{x_n\} = \lim_{n\to\infty} F\{y_n\} = t = F(w)$$

We claim that $H(w) = F(w)$, if it is not then we put $x = w$ and $y = x_n$ in equation (1).

$$\|Hw - Lx_n\| \leq \frac{k_1}{2} max\{\|Gx_n - Lw\|, \|Hw - Lx_n\|, \|Lx_n - Gx_n\|\}$$
$$+ \frac{k_2}{2}\{\|Lw - Lx_n\|, \|Lx_n - Hw\|\}$$

Taking Limit $n \to \infty$, we get

$$\|Hw - t\| \leq \frac{k_1}{2} max\{\|t - t\|, \|Hw - t\|, \|t - t\|\} + \frac{k_2}{2}\{\|t - t\|, \|t - Hw\|\}$$

$$\left(1 - \frac{k_1}{2} - \frac{k_2}{2}\right)\|Hw - t\| = 0$$

$$H(w) = t$$

$$F(w) = H(w) = t$$

Hence w is coincidence point of(H, F).

Now from the weak computability of(F, H) we have

$HF(w) = FH(w)$ or $Ht = Ft$.

Since $H(X) \subseteq G(X)$ there is an element $z \in X$. such that $H(w) = G(z)$.

Thus $H(w) = F(w) = G(z) = t$

We show that z is coincidence point of (L, G) is z, that is $G(z) = L(z) = t$

If not then we put $x = w$ and $y = z$ in equation (3.1)

$$\|Hw - Lz\| \leq \frac{k_1}{2} max\{\|Gz - Lw\|, \|Hw - Lz\|, \|Lz - Gz\|\} + \frac{k_2}{2}\{\|Lw - Lz\|, \|Lz - Hw\|\}$$

Taking Limit $n \to \infty$, we get

$$\|t - Lz\| \leq \frac{k_1}{2}\{\|t - t\|, \|t - L(z)\|, \|t - Lz\|\} + \frac{k_2}{2}\{\|t - Lz\|, \|t\ (z) - Hw\|\}$$

$$\|t - Lz\| \leq \frac{k_1}{2}\{\|t - L(z)\|\} + k_2$$

$$\left(1-\frac{k_1}{2}-\frac{k_2}{2}\right)\|t-L(z)\| \leq 0$$

$$t = L(z)$$

Clearly $L(z) = G(z) = t$,

z is a coincidence point of (L, G). Since the pair (L, G) are weak compatible

$$\Longrightarrow GL(w) = LG(w) \; or \; Lt = Gt$$

Hence F, G, H and L have a common coincidence point$t$.

Next, we prove that the common fixed point of F, G, H and L. So we put that $x = w$ and $y = t$ in equation (1)

$$\|Hw - Lt\| \leq \frac{k_1}{2} max\{\|Gt - Fw\|, \|Hw - Lt\|, \|Lt - Gt\|\} + \frac{k_2}{2}\{\|Lw - Lt\|, \|Lt - Hw\|\}$$

$$\|t - L(t)\| \leq \frac{k_1}{2} max\{\|Lt - t\|, \|t - Lt\|, \|t - t\|\} + \frac{k_2}{2}\{\|t - Lt\|, \|t - Lt\|\}$$

$$\|t - Lt\| \leq 0 \Longrightarrow t = L(t)$$

Clearly $F(t) = H(t) = L(t) = G(t) = t$

Hence t is common fixed point of F, H, G and L.

Let t' be another fixed point of F, G, H and L

We assume that $G(X)$ is a complete subspace of X, a similar argument obtains. If the pair (H, F) satisfies property (E.A.) then we get similar result.

Fixed Point Theorem Using CLR Property:

Theorem 2: Two self-mappings F , G , $H$and Lbe defined on Cone Banach Space $(X, \|.\|)$ with $\|x\| = d(x, 0)$ satisfying the condition

$$\|Hx - Ly\| \leq \frac{k_1}{2} \max\{\|Gy - Fx\|, \|Hx - Ly\|, \|Ly - Gy\|\} +$$

$$\frac{k_2}{2}\{\|Fx - Ly\|, \|Ly - Hx\|\} \qquad (3.2)$$

Where k_1 and k_2 are non-negative and $\left(1-\frac{k_1}{2}-k_2\right)<1$

(i) $H(X) \subseteq G(X)$ and $L(X) \subseteq F(X)$

(ii) The pair $(H,F)$and(L,G) are weakly compatible.

(iii) The pair (L,G)or (H,F)satisfied by (CLR_L) and (CLR_H) Property.

Then F , G , H and Lhave a unique common fixed point.

Proof: First we assume that the pair (L,G) satisfied the (CLR_L) Property then there exist the sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Lx_n = \lim_{n\to\infty} Gx_n = Lx \text{ for some } x \in X$$

Further, since $L(X) \subseteq F(X)$We have $Lx = Fw$ for some $w \in X$.

We claim that

$Hw = Fw = t$ (say). If not then $x = w$ and $y = x_n$ in (1)

$$\|Hw - Lx_n\| \le \frac{k_1}{2} max\,\{\|Gx_n - Fw\|, \|Hw - Lx_n\|, \|Lx_n - Gx_n\|\}$$
$$+\frac{k_2}{2}\{\|Fw - Lx_n\|, \|Lx_n - Hw\|\}$$

$$\|Hw - Lx_n\| \le \frac{k_1}{2} max\,\{\|Lx - Lx\|, \|Hw - Lx\|, \|Lx - Lx\|\} + \frac{k_2}{2}\{\|Lx - Lx\|, \|Lx - Hw\|\}$$

$$\|Hw - Lx_n\| \le \frac{k_1}{2}\|Hw - Lx\| + \frac{k_2}{2}\|Lx - Hw\|$$

$$\|Hw - Lx_n\|\left(1 - \frac{k_1}{2} - \frac{k_2}{2}\right) \le 0$$

$$H(w) = L(x)$$

Hence$Hw = Lx$ implies that$Fw = Hw = Lx = t$

Hence w is a coincidence point of H and F.

Since the pair (H,F) is weak compatible

$$\Longrightarrow HFw = Fhw = Ht = Ft$$

Further Since $H(X) \subseteq G(M)$, there exist some $z \in X$

Such that $H(w) = G(z)$

We claim that $L(z) = t$

On the contradiction we put, $x = w$ and $y = z$ in equation (3.2)

$$\|Hw - Lz\| \leq \frac{k_1}{2} max\,\{\|Gz - Fw\|, \|Hw - Lz\|, \|Lz - Gz\|\} + \frac{k_2}{2}\{\|Fw - Lz\|, \|Lz - Hw\|\}$$

$$\|Hw - Lz\| \leq \frac{k_1}{2} max\,\{\|Hw - Hw\|, \|Hw - Lz\|, \|Lz - Hw\|\} + \frac{k_2}{2}\{\|Hw - Lz\|, \|Lz - Hw\|\}$$

$$\|Hw - Lz\| \leq \frac{k_1}{2}\|Hw - Lz\| + k_2\|Hw - Lz\|$$

$$\left(1 - \frac{k_1}{2} - k_2\right)\|Hw - Lz\| \leq 0$$

$$H(w) = L(z)$$

$$\Longrightarrow t = L(z)$$

Hence $Lz = t$, hence $Fw = Hw = Lz = Gz = t$

It shows that z is coincidences point G.

Also the weak compatibility of (L, G) implies that

$$LGz = Glz = Lt = Gt$$

$x = w$ and $y = t$ in equation (1).

$$\|Hw - Lt\| \leq \frac{k_1}{2} max\,\{\|Gw - Fw\|, \|Hw - Lt\|, \|Lt - Gt\|\} + \frac{k_2}{2}\{\|Fw - Lt\|, \|Lt - Hw\|\}$$

$$\|t - Lt\| \leq \frac{k_1}{2} max\,\{0, \|t - Lt\|, \|Lt - t\|\} + \frac{k_2}{2}\{\|t - Lt\|, \|Lt - t\|\}$$

$$\left(1 - \frac{k_1}{2} - \frac{k_2}{2}\right)\|t - Lt\| \leq 0$$

$$t = Lt$$

Hence $Ft = Ht = Lt = Gt = t$.

It shows that t is a common fixed point of F , G , H and L. Easily, we determine uniqueness.

Similarly, the argument that the pair (H, F) satisfy the (CLR_H).The property will also give the unique common fixed point of F , G , H and L.

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