Exact Solution of System of Linear Fredholm Fractional Integro-Differential Equations using Direct Computation Method

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Abstract

The goal of the current study is to develop a method and solve a system of linear Fredholm fractional integro-differential equations involving the Caputo derivative. In order to solve a system of linear Fredholm fractional integro-differential equations, the direct computation method has been constructed. The advantage of the method is to get the solution in exact form rather than in series form. The exact solution provides a complete and precise answer to the problem. In this study, exact solutions are obtained for the system of linear Fredholm fractional integro-differential equations. The developed method is well illustrated and solutions obtained are simulated using Scilab 6.1.1.

Keywords: Fractional integro-differential equations, Exact solution, Caputo derivative, Direct computation method.
Subject Classification: 26A33, 34A08

1 Introduction

The generalization of the integer order integro-differential equations is the fractional order integro-differential equations (FIDEs). The motivation to extend integer order integro-differential equations to fractional order integro-differential equations comes from the need to model systems that exhibit complex behavior and cannot be accurately described by traditional integer-order models. Fractional integro-differential equations can result from modelling processes in scientific fields like finance, biology, physics, and engineering. The system of fractional integro-differential equations was solved using various analytical and numerical techniques, to find both approximate and exact solutions. Compared to traditional numerical methods, our method gives the exact solution and is free from round-off errors.
Several numerical methods to solve system of FIDEs are least square method with the aid of Chebyshev polynomials [5], least square method with the aid of Hermite polynomials [6], Taylor expansion [2], Decomposition method [8], Homotopy perturbation method [13], Adomian decomposition method [14], Chebyshev spectral method [18], and Homotopy analysis method (HAM) [19].
Ebaid et al. [3] study a straight forward analytical approach for getting precise answers for the class of first order FIDEs in terms of Caputo derivative. Fadi et al. [1] applied the homotopy analysis method to provide an analytical solution for the fractional integro-differential equations. A novel numerical technique for evaluating a linear system of FIDEs has been provided by Mahdy et al. in the paper [5]. By employing Hermite polynomials and the least square method, A. M. S. Mahdy [6] proposed a numerical method to solve the linear system of FIDEs. According to the Didgar et al. [2], a new use of the Taylor expansion is presented as an approximation method for solving linear systems of FIDEs. Momani et al. [8] utilized the decomposition method to implement an approximate solution of the FIDEs system, and the fractional derivative is interpreted in the Caputo sense. Nanware et al. [9] used Bernstein polynomials to obtain the numerical solution of FIDEs, whereas [10] enhanced the outcomes by using Hermite polynomials for even lower n values. Nanware et al. [11] solved an exact solution of both linear and non-linear Fredholm FIDEs using a

direct computation method.
A homotopy perturbation method was used by Rostam K. S. and Hassan M. Sdeq in the study [13] to generate the solution to the system of Fredholm FIDEs. Saleh et al. [14] employed the Adomian decomposition approach to compute a system of linear FIDEs. Systems of fractional integro-differential equations and Abel's integral equations are both solved using the Chebyshev spectral method, which is based on an operational matrix [18]. A new analytical method to solve systems of FIDEs was proposed by Mohammad Zurigat et al. [19] and is based on the Homotopy analysis method (HAM). The work demonstrates the applicability and enormous prospective of HAM for solving linear and nonlinear fractional integro-differential systems of equations. The approximate controllability of Hilfer fractional neutral Volterra integro-differential inclusions via almost sectorial operators was proposed by Bose et al. [16]. The results are demonstrated with the approximate controllability of the fractional system. The existence of Hilfer fractional neutral stochastic Volterra integro-differential inclusions with almost sectorial operators was proposed by Sivasankar et al. [15]. The results of the problems are proved using fractional calculus, stochastic analysis theory, and the fixed point theorem for multivalued maps.
Aforementioned literature motivates us to consider the system of linear Fredholm FIDEs and our aim is to develop a method to obtain exact solution of the following systems of linear Fredholm FIDEs [17] of second kind:

$$\begin{aligned} {}^cD^{\nu}w_1(x) &= F_1(x) + \int_a^b \Big[K_1(x,t)w_1(t) + \tilde{K}_1(x,t)w_2(t)\Big]\,dt, \\ {}^cD^{\nu}w_2(x) &= F_2(x) + \int_a^b \Big[K_2(x,t)w_1(t) + \tilde{K}_2(x,t)w_2(t)\Big]\,dt, \end{aligned} \tag{1}$$

$$w_n^{(j)}(x_0) = w_{nj}, \quad n = 1,2,...,i, \quad j = 0,1,...,n-1, \quad n-1 < \nu \le n, \quad n \in N. \tag{2}$$

where ${}^cD^{\nu}w_1(x)$ and ${}^cD^{\nu}w_2(x)$ represents ν^{th} Caputo fractional derivative of $w_1(x)$ and $w_2(x)$. $F_1(x)$, $F_2(x)$ and $K(x,t)$ are given functions, with real variables x and t ranging between $[0,1]$.
In this work we constructed direct computation method for the problem (1)-(2). We organize the paper in the following way: Section 2 recalls preliminaries, basic definitions related to fractional calculus and some properties. In Section 3, Direct computation method is constructed for solving systems of linear Fredholm FIDEs. Section 4, presents an analytical examples to illustrate the accuracy of our constructed method. The final Section 5 gives the conclusion.

2 Preliminaries

We present few basic definitions for fractional calculus and put forward several properties that can be used to find solutions to the problems.
Definition 2.1 [7] A real function $w(r)$, $r > 0$ is said to be in the space C_μ, $\mu \in R$, if there exists a real number $p > \mu$ such that $w(r) = r^p w_1(r)$, where $w_1(r) \in C[0,\infty)$ and it is said to be in the space C_μ^n if and only if $w^n \in C_\mu$, $n \in N$.
Definition 2.2 [12] The Riemann-Liouville fractional integral of order $\nu > 0$ for a function $w(r)$ is defined as:

$$I^{\nu}w(r) = \frac{1}{\Gamma(\nu)}\int_0^r (r-t)^{(\nu-1)}w(t)dt, \quad r > 0$$

Definition 2.3 [12] The Caputo fractional derivative of a function $w(r)$ of order ν is denoted by ${}^cD^{\nu}$ and is defined as:

$${}^cD_r^{\nu}w(r) = \begin{cases} \frac{1}{\Gamma(n-\nu)}\int_0^r (r-t)^{n-\nu-1}\frac{d^n w(t)}{dt^n}dt, & n-1 < \nu < n \\ \frac{d^n w(r)}{dr^n}, \quad \nu = n, \quad n \in N. \end{cases}$$

We have the following properties:

(i) $I^{\nu}I^{\phi}w(r) = I^{\nu+\phi}w(r), \quad \nu,\phi > 0, \quad w \in C_\mu, \quad \mu > 0.$

(ii) $I^{\nu}{}^cD^{\nu}w(r) = w(r) - \sum_{k=0}^{n-1} w^k(0^+)\frac{r^k}{k}, \quad r > 0, \quad n-1 < \nu \le n.$

(iii) $I^{\nu}r^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1+\nu)}r^{\beta+\nu}, \quad \nu > 0, \quad \beta > -1, \quad r > 0.$

(iv) ${}^cD^\nu I^\nu w(r) = w(r), \quad r > 0, \quad n-1 < \nu \leq n.$

(v) ${}^cD^\nu r^\beta = -\begin{cases} 0, & \beta \in N_0, \quad \beta < \lceil \nu \rceil \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta-\nu+1)} r^{\beta-\nu}, & \beta \in N_0, \quad and \quad \beta \geq \lceil \nu \rceil \end{cases}$

(vi) ${}^cD^\nu$ A=0, A represents constant.

where $\lceil \nu \rceil$ denoted the smallest integer greater than or equal to ν and $N_0 = \{0, 1, 2, ...\}$

3 Analysis of the Direct Computation Method

In this section, the system of linear Fredholm fractional integro-differential equations will be solved directly. The strategy is straight forward and presents an exact solution to the system of linear Fredholm FIDEs. It should be noted that this procedure will be used for the degenerate kernels.

Definition 3.1[4] A kernel $K(x,t)$ that can be written as the sum of a finite number of terms, each of which is the product of a function of x and a function of t, is said to be degenerate kernel.

$$\begin{aligned} K_1(x,t) &= \sum_{i=1}^{m} f_i(x)h_i(t), \quad \tilde{K}_1(x,t) = \sum_{i=1}^{m} \tilde{f}_i(x)\tilde{h}_i(t), \\ K_2(x,t) &= \sum_{i=1}^{m} p_i(x)q_i(t), \quad \tilde{K}_2(x,t) = \sum_{i=1}^{m} \tilde{p}_i(x)\tilde{q}_i(t). \end{aligned} \tag{3}$$

Substituting (3) into (1) gives

$$\begin{aligned} {}^cD^\nu w_1(x) &= F_1(x) + \int_a^b \left(\sum_{i=1}^{m} f_i(x)h_i(t)w_1(t) + \sum_{i=1}^{m} \tilde{f}_i(x)\tilde{h}_i(t)w_2(t) \right) dt, \\ {}^cD^\nu w_2(x) &= F_2(x) + \int_a^b \left(\sum_{i=1}^{m} p_i(x)q_i(t)w_1(t) + \sum_{i=1}^{m} \tilde{p}_i(x)\tilde{q}_i(t)w_2(t) \right) dt. \end{aligned}$$

This implies

$$\begin{aligned} {}^cD^\nu w_1(x) &= F_1(x) + \sum_{i=1}^{m} f_i(x) \left(\int_a^b h_i(t)w_1(t) \right) dt + \sum_{i=1}^{m} \tilde{f}_i(x) \left(\int_a^b \tilde{h}_i(t)w_2(t) \right) dt, \\ {}^cD^\nu w_2(x) &= F_2(x) + \sum_{i=1}^{m} p_i(x) \left(\int_a^b q_i(t)w_1(t) \right) dt + \sum_{i=1}^{m} \tilde{p}_i(x) \left(\int_a^b \tilde{q}_i(t)w_2(t) \right) dt. \end{aligned} \tag{4}$$

The right side of the above equations, every integral rely only on the variable $'t'$ having limits constant.
Equation (4) changes to,

$$\begin{aligned} {}^cD^\nu w_1(x) &= F_1(x) + \alpha_1 f_1(x) + \alpha_2 f_2(x) + ... + \alpha_m f_m(x) + \beta_1 \tilde{f}_1(x) + ... + \beta_m \tilde{f}_m(x), \\ {}^cD^\nu w_2(x) &= F_2(x) + \gamma_1 p_1(x) + \gamma_2 p_2(x) + ... + \gamma_m p_m(x) + \delta_1 \tilde{p}_1(x) + ... + \delta_m \tilde{p}_m(x). \end{aligned} \tag{5}$$

where

$$\begin{aligned} \alpha_i &= \int_a^b h_i(t)w_1(t)dt, \\ \beta_i &= \int_a^b \tilde{h}_i(t)w_2(t)dt, \\ \gamma_i &= \int_a^b q_i(t)w_1(t)dt, \\ \delta_i &= \int_a^b \tilde{q}_i(t)w_2(t)dt. \end{aligned} \tag{6}$$

Integrating both sides of equation (5) of order ν times starting 0 to x, by utilising the assigned conditions, we get

$$\begin{aligned} w_1(x) - \sum_{m=0}^{n-1} \frac{w_1^{(m)}(0)}{m!} x^m(0) &= I^\nu [F_1(x)] + \alpha_1 I^\nu [f_1(x)] + \alpha_2 I^\nu [f_2(x)] + ... + \alpha_m I^\nu [f_m(x)] + \\ &\beta_1 I^\nu \left[\tilde{f}_1(x)\right] + \beta_2 I^\nu \left[\tilde{f}_2(x)\right] ... + \beta_m I^\nu \left[\tilde{f}_m(x)\right], \\ w_2(x) - \sum_{m=0}^{n-1} \frac{w_2^{(m)}(0)}{m!} x^m(0) &= I^\nu [F_2(x)] + \gamma_1 I^\nu [p_1(x)] + \gamma_2 I^\nu [p_2(x)] + ... + \gamma_m I^\nu [p_m(x)] + \\ &\delta_1 I^\nu [\tilde{p}_1(x)] + \delta_2 I^\nu [\tilde{p}_2(x)] ... + \delta_m I^\nu [\tilde{p}_m(x)]. \end{aligned} \tag{7}$$

A system of algebriac equations is produced by solving the above mentioned equations for $w_1(x)$ and $w_2(x)$, which are computed to get the constants α_i, β_i, γ_i & δ_i. The solutions for system of linear Fredholm FIDEs are found by inserting the calculated constant values into the deduced equations $w_1(x)$ & $w_2(x)$.

4 Illustrative Examples

We choose a system of linear Fredholm FIDEs to illustrate the proposed technique.

Example 4.1 *Consider the system of Fredholm fractional integro-differential equation [14]*

$$\begin{aligned} D^{\frac{2}{3}} w_1(x) &= \frac{3x^{\frac{1}{3}}\sqrt(3)\Gamma(2/3)}{2\pi} - \frac{x}{6} + \int_0^1 2xt\,[w_1(t) + w_2(t)]\,dt, \\ D^{\frac{2}{3}} w_2(x) &= \frac{9x^{\frac{4}{3}}\sqrt(3)\Gamma(2/3)}{4\pi} + \frac{5x^3}{6} + \int_0^1 x^3\,[w_1(t) - w_2(t)]\,dt, \end{aligned} \tag{8}$$

subject to $w_1(0) = -1$, $w_2(0) = 0$.
Let

$$\alpha = \int_0^1 t\,[w_1(t) + w_2(t)]\,dt, \tag{9}$$

$$\beta = \int_0^1 [w_1(t) - w_2(t)]\,dt. \tag{10}$$

where α *and* β *are unknown constants. Accordingly, equations* (8) *becomes,*

$$D^{\frac{2}{3}} w_1(x) = \frac{3x^{\frac{1}{3}}\sqrt(3)\Gamma(2/3)}{2\pi} - \frac{x}{6} + 2x\alpha,$$

$$D^{\frac{2}{3}} w_1(x) = \frac{3x^{\frac{1}{3}}\sqrt(3)\Gamma(2/3)}{2\pi} + \left(2\alpha - \frac{1}{6}\right) x. \tag{11}$$

and

$$D^{\frac{2}{3}} w_2(x) = \frac{9x^{\frac{4}{3}}\sqrt(3)\Gamma(2/3)}{4\pi} + \frac{5x^3}{6} + x^3\beta,$$

$$D^{\frac{2}{3}} w_2(x) = \frac{9x^{\frac{4}{3}}\sqrt(3)\Gamma(2/3)}{4\pi} + \left(\frac{5}{6} + \beta\right) x^3. \tag{12}$$

Operating $I^{\frac{2}{3}}$ *on both sides of the equations* (11) *and* (12),

$$w_1(x) = -1 + x + \left(2\alpha - \frac{1}{6}\right) \frac{x^{5/3}}{\Gamma(8/3)}, \tag{13}$$

$$w_2(x) = x^2 + \left(\frac{5}{6} + \beta\right)\frac{6x^{11/3}}{\Gamma(14/3)}. \tag{14}$$

Substitute equation (13) *and* (14) *in the equation* (9) *and* (10), *then equation becomes*

$$\alpha = \int_0^1 t\left[t - 1 + \left(2\alpha - \frac{1}{6}\right)\frac{t^{5/3}}{\Gamma(8/3)} + t^2 + \left(\frac{5}{6} + \beta\right)\frac{6t^{11/3}}{\Gamma(14/3)}\right] dt,$$

$$\beta = \int_0^1 \left[t - 1 + \left(2\alpha - \frac{1}{6}\right)\frac{t^{5/3}}{\Gamma(8/3)} - t^2 - \left(\frac{5}{6} + \beta\right)\frac{6t^{11/3}}{\Gamma(14/3)}\right] dt.$$

Solving the above equations we get system of equations as,

$$\alpha\left[1 - \frac{6}{11\Gamma(8/3)}\right] + \beta\left[\frac{-18}{17\Gamma(14/3)}\right] = \frac{1}{12} - \frac{3}{66\Gamma(8/3)} + \frac{15}{17\Gamma(14/3)}, \tag{15}$$

$$\alpha\left[-\frac{6}{8\Gamma(8/3)}\right] + \beta\left[1 + \frac{18}{14\Gamma(14/3)}\right] = \frac{-5}{6} - \frac{3}{48\Gamma(8/3)} - \frac{15}{14\Gamma(14/3)}. \tag{16}$$

Solving the above equations (15) *and* (16)*from using Scilab software, we get*

$$\alpha = \frac{1}{12} \quad and \quad \beta = -\frac{5}{6}.$$

Substitute α *and* β *in equation* (13) *and* (14) *we get exact solution of* $w_1(x)$ *and* $w_2(x)$*as,*

$$w_1(x) = x - 1 \quad and \quad w_2(x) = x^2.$$

Figure below depicts the computational results for the system of Fredholm FIDEs (8).

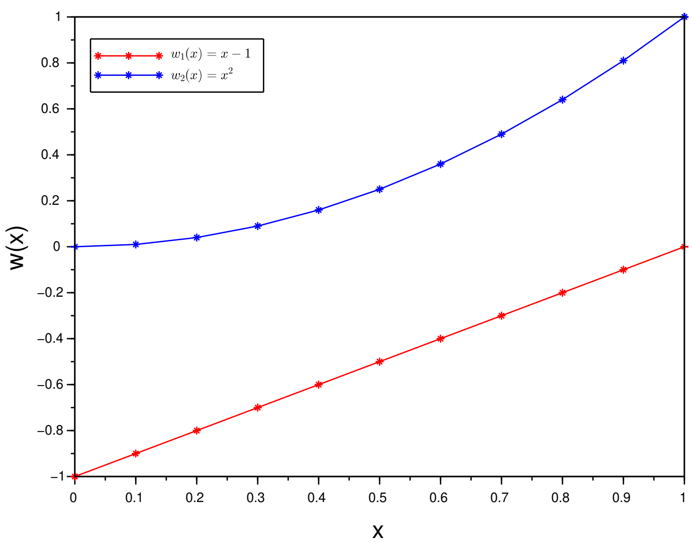

Figure 1: Solution of System (8)

Example 4.2 *Consider the system of Fredholm fractional integro-differential equation [14]*

$$D^{\frac{3}{4}} w_1(x) = \frac{2x^{\frac{1}{4}}\sqrt{(2)}\Gamma(3/4)(15 - 32x^2)}{15\pi} - \frac{1}{20} - \frac{x}{12} + \int_0^1 (x+t)\left[w_1(t) + w_2(t)\right] dt,$$

$$D^{\frac{3}{4}} w_2(x) = \frac{2x^{\frac{1}{4}}\sqrt{(2)}\Gamma(3/4)(8x - 5)}{5\pi} - \frac{2\sqrt{(x)}}{15} + \int_0^1 \sqrt{(x)}t^2\left[w_1(t) - w_2(t)\right] dt, \tag{17}$$

subject to $w_1(0) = 0,\ w_2(0) = 0.$
Let

$$\begin{aligned} \alpha &= \int_0^1 [w_1(t) + w_2(t)]\, dt, \\ \beta &= \int_0^1 t\, [w_1(t) + w_2(t)]\, dt, \\ \gamma &= \int_0^1 t^2\, [w_1(t) - w_2(t)]\, dt. \end{aligned} \tag{18}$$

where α , β *and* γ *are unknown constants. Accordingly, equation* (17) *becomes,*

$$\begin{aligned} D^{\frac{3}{4}} w_1(x) &= \frac{2x^{\frac{1}{4}}\sqrt{(2)}\Gamma(3/4)}{\pi} - \frac{64x^{\frac{9}{4}}\sqrt{(2)}\Gamma(3/4)}{15\pi} + \left(\beta - \frac{1}{20}\right) + \left(\alpha - \frac{1}{12}\right) x, \\ D^{\frac{3}{4}} w_2(x) &= \frac{16x^{\frac{5}{4}}\sqrt{(2)}\Gamma(3/4)}{5\pi} - \frac{10x^{\frac{1}{4}}\sqrt{(2)}\Gamma(3/4)}{5\pi} + \left(\gamma - \frac{2}{15}\right) x^{\frac{1}{2}}. \end{aligned} \tag{19}$$

Operating $I^{\frac{3}{4}}$ *on both sides of the equations* (19),

$$\begin{aligned} w_1(x) &= x - x^3 + \left(\beta - \frac{1}{20}\right)\frac{x^{\frac{3}{4}}}{\Gamma(7/4)} + \left(\alpha - \frac{1}{12}\right)\frac{x^{\frac{7}{4}}}{\Gamma(11/4)}, \\ w_2(x) &= x^2 - x + \left(\gamma - \frac{2}{15}\right)\frac{\Gamma(3/2)x^{\frac{5}{4}}}{\Gamma(9/4)}. \end{aligned} \tag{20}$$

Substitute equation (20) *in the equation* (18), *then equation* (18) *becomes*

$$\begin{aligned} \alpha &= \int_0^1 \left[t - t^3 + \left(\beta - \frac{1}{20}\right)\frac{t^{\frac{3}{4}}}{\Gamma(7/4)} + \left(\alpha - \frac{1}{12}\right)\frac{t^{\frac{7}{4}}}{\Gamma(11/4)} + t^2 - t + \left(\gamma - \frac{2}{15}\right)\frac{\Gamma(3/2)t^{\frac{5}{4}}}{\Gamma(9/4)}\right] dt, \\ \beta &= \int_0^1 t\left[t - t^3 + \left(\beta - \frac{1}{20}\right)\frac{t^{\frac{3}{4}}}{\Gamma(7/4)} + \left(\alpha - \frac{1}{12}\right)\frac{t^{\frac{7}{4}}}{\Gamma(11/4)} + t^2 - t + \left(\gamma - \frac{2}{15}\right)\frac{\Gamma(3/2)t^{\frac{5}{4}}}{\Gamma(9/4)}\right] dt, \\ \gamma &= \int_0^1 t^2\left[t - t^3 + \left(\beta - \frac{1}{20}\right)\frac{t^{\frac{3}{4}}}{\Gamma(7/4)} + \left(\alpha - \frac{1}{12}\right)\frac{t^{\frac{7}{4}}}{\Gamma(11/4)} - t^2 + t - \left(\gamma - \frac{2}{15}\right)\frac{\Gamma(3/2)t^{\frac{5}{4}}}{\Gamma(9/4)}\right] dt. \end{aligned}$$

Solving the above three equations we get three system of equations as,

$$\begin{aligned} \alpha\left[1 - \frac{4}{11\Gamma(11/4)}\right] + \beta\left[\frac{-4}{7\Gamma(7/4)}\right] + \gamma\left[\frac{-4\Gamma(3/2)}{9\Gamma(9/4)}\right] &= \frac{1}{12} - \frac{4}{140\Gamma(7/4)} - \frac{4}{132\Gamma(11/4)} - \frac{8\Gamma(3/2)}{135\Gamma(9/4)}, \\ \alpha\left[\frac{-4}{15\Gamma(11/4)}\right] + \beta\left[1 - \frac{4}{11\Gamma(7/4)}\right] + \gamma\left[\frac{-4\Gamma(3/2)}{13\Gamma(9/4)}\right] &= \frac{1}{20} - \frac{4}{220\Gamma(7/4)} - \frac{4}{180\Gamma(11/4)} - \frac{8\Gamma(3/2)}{195\Gamma(9/4)}, \\ \alpha\left[\frac{-4}{19\Gamma(11/4)}\right] + \beta\left[\frac{-4}{15\Gamma(7/4)}\right] + \gamma\left[1 + \frac{4\Gamma(3/2)}{17\Gamma(9/4)}\right] &= \frac{2}{15} - \frac{4}{300\Gamma(7/4)} - \frac{4}{228\Gamma(11/4)} + \frac{8\Gamma(3/2)}{255\Gamma(9/4)}. \end{aligned} \tag{21}$$

Solving the above system of equtions (21) *we get*

$$\alpha = \frac{1}{12}, \quad \beta = \frac{1}{20} \quad \text{and} \quad \gamma = \frac{2}{15}.$$

Substitute α , β *and* γ *in equation* (20) *we get exact solution of* $w_1(x)$ *and* $w_2(x)$*as,*

$$w_1(x) = x - x^3 \quad \text{and} \quad w_2(x) = x^2 - x.$$

Figure below depicts the computational results for the system of Fredholm FIDEs (17).

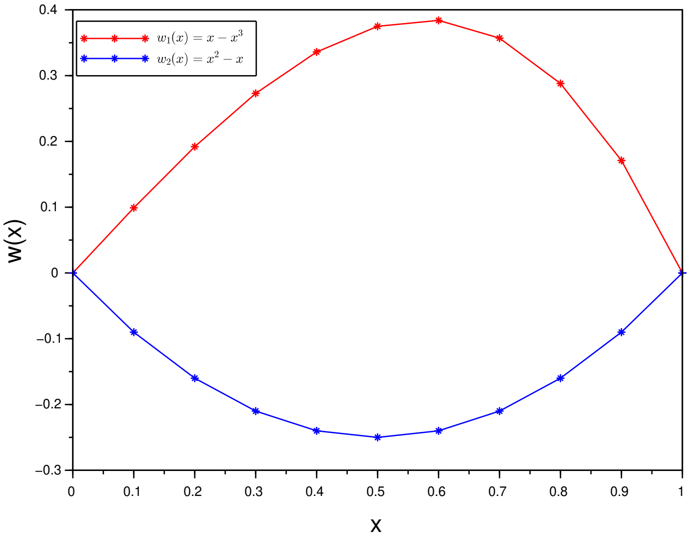

Figure 2: Solution of System (17)

Example 4.3 *Consider the system of Fredholm fractional integro-differential equation [2]*

$$\begin{aligned} D^{\frac{4}{5}} w_1(x) &= \frac{83x}{80} - \frac{25x^{\frac{6}{5}}}{3\Gamma(1/5)} + \frac{125x^{\frac{11}{5}}}{11\Gamma(1/5)} + \int_0^1 2xt\,[w_1(t) - w_2(t)]\,dt, \\ D^{\frac{4}{5}} w_2(x) &= \frac{-67}{160} - \frac{13x}{24} + \frac{125x^{\frac{6}{5}}}{8\Gamma(1/5)} + \int_0^1 (x+t)\,[w_1(t) + w_2(t)]\,dt, \end{aligned} \tag{22}$$

subject to $w_1(0) = 0,\ w_2(0) = 0.$
Let

$$\begin{aligned} \alpha &= \int_0^1 t\,[w_1(t) - w_2(t)]\,dt, \\ \beta &= \int_0^1 [w_1(t) + w_2(t)]\,dt, \\ \gamma &= \int_0^1 t\,[w_1(t) + w_2(t)]\,dt. \end{aligned} \tag{23}$$

where α , β *and* γ *are unknown constants. Accordingly, equation* (22) *becomes,*

$$\begin{aligned} D^{\frac{4}{5}} w_1(x) &= -\frac{25x^{\frac{6}{5}}}{3\Gamma(1/5)} + \frac{125x^{\frac{11}{5}}}{11\Gamma(1/5)} + \left(2\alpha + \frac{83}{80}\right) x, \\ D^{\frac{4}{5}} w_2(x) &= \frac{125x^{\frac{6}{5}}}{8\Gamma(1/5)} + \left(\beta - \frac{13}{24}\right) x + \left(\gamma - \frac{67}{160}\right). \end{aligned} \tag{24}$$

Operating $I^{\frac{4}{5}}$ *on both sides of the equations* (24),

$$\begin{aligned} w_1(x) &= x^3 - x^2 + \left(2\alpha + \frac{83}{80}\right) \frac{x^{\frac{9}{5}}}{\Gamma(14/5)}, \\ w_2(x) &= \frac{15x^2}{8} + \left(\beta - \frac{13}{24}\right) \frac{x^{\frac{9}{5}}}{\Gamma(14/5)} + \left(\gamma - \frac{67}{160}\right) \frac{x^{\frac{4}{5}}}{\Gamma(9/5)}. \end{aligned} \tag{25}$$

Substitute equation (25) *in the equation* (23), *then equation* (23) *becomes*

$$\alpha = \int_0^1 t\left[t^3 - t^2 + \left(2\alpha + \frac{83}{80}\right)\frac{t^{\frac{9}{5}}}{\Gamma(14/5)} - \frac{15t^2}{8} - \left(\beta - \frac{13}{24}\right)\frac{t^{\frac{9}{5}}}{\Gamma(14/5)} - \left(\gamma - \frac{67}{160}\right)\frac{t^{\frac{4}{5}}}{\Gamma(9/5)}\right]dt,$$

$$\beta = \int_0^1 \left[t^3 - t^2 + \left(2\alpha + \frac{83}{80}\right)\frac{t^{\frac{9}{5}}}{\Gamma(14/5)} + \frac{15t^2}{8} + \left(\beta - \frac{13}{24}\right)\frac{t^{\frac{9}{5}}}{\Gamma(14/5)} + \left(\gamma - \frac{67}{160}\right)\frac{t^{\frac{4}{5}}}{\Gamma(9/5)}\right]dt,$$

$$\gamma = \int_0^1 t\left[t^3 - t^2 + \left(2\alpha + \frac{83}{80}\right)\frac{t^{\frac{9}{5}}}{\Gamma(14/5)} + \frac{15t^2}{8} + \left(\beta - \frac{13}{24}\right)\frac{t^{\frac{9}{5}}}{\Gamma(14/5)} + \left(\gamma - \frac{67}{160}\right)\frac{t^{\frac{4}{5}}}{\Gamma(9/5)}\right]dt.$$

Solving the above three equations we get three system of equations as,

$$\begin{aligned}
\alpha\left[1 - \frac{10}{19\Gamma(14/5)}\right] + \beta\left[\frac{5}{19\Gamma(14/5)}\right] + \gamma\left[\frac{5}{14\Gamma(9/5)}\right] &= \frac{-83}{160} + \frac{415}{1520\Gamma(14/5)} + \frac{65}{456\Gamma(14/5)} + \frac{335}{2240\Gamma(9/5)},\\
\alpha\left[\frac{-10}{14\Gamma(14/5)}\right] + \beta\left[1 - \frac{5}{14\Gamma(14/5)}\right] + \gamma\left[\frac{-5}{9\Gamma(9/5)}\right] &= \frac{13}{24} + \frac{415}{1120\Gamma(14/5)} - \frac{65}{336\Gamma(14/5)} - \frac{335}{1440\Gamma(9/5)},\\
\alpha\left[\frac{-10}{19\Gamma(14/5)}\right] + \beta\left[\frac{-5}{19\Gamma(14/5)}\right] + \gamma\left[1 - \frac{5}{14\Gamma(9/5)}\right] &= \frac{67}{160} + \frac{415}{1520\Gamma(14/5)} - \frac{65}{456\Gamma(14/5)} - \frac{335}{2240\Gamma(9/5)}.
\end{aligned} \tag{26}$$

Solving the above system of equtions (26) *we get*

$$\alpha = -0.51875, \quad \beta = \frac{13}{24} \quad \text{and} \quad \gamma = \frac{67}{160}.$$

Substitute α, β *and* γ *in equation* (25) *we get exact solution of* $w_1(x)$ *and* $w_2(x)$ *as,*

$$w_1(x) = x^3 - x^2 \quad \text{and} \quad w_2(x) = \frac{15x^2}{8}.$$

Figure below depicts the computational results for the system of Fredholm FIDEs (22).

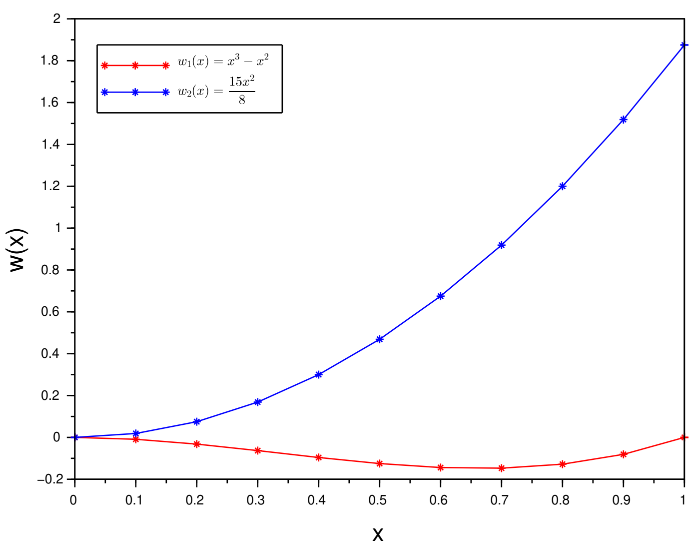

Figure 3: Solution of System (22)

5 Conclusion

The direct computation method is developed for the system of linear Fredholm FIDEs. The method is well illustrated with examples when the kernel is degenerate. The obtained exact solutions are simulated using Scilab 6.1.1 software and the results are graphically represented. The method requires computation of co-coefficients involved in the system of linear equations is difficult for number of equations are greater than 2. In future the constructed method can be extended and generalized for different types of kernels in the system of linear Fredholm FIDEs.

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