# PATHWAY FRACTIONAL INTEGRAL OPERATOR WITH COMPOSITION OF GENERALIZED FUNCTION $G_{\rho,\eta,r}[a,z]$

*Seema Kabra*[a] , *Omprakash Saini*[b] *and Deepak Kumar Kabra*[a*]

[a] *Department of Mathematics*
*Sangam University, Bhilwara-311001, Rajasthan, India*
[b] *Research Scholar,*
*Sangam University, Bhilwara-311001, Rajasthan, India*
*e-mails: kabraseema@rediffmail.com, opsainisikar@gmail.com and dkabra20@gmail.com*
* ***corresponding author***

The purpose of this paper is to consider the properties of generalized function $G_{\rho,\eta,r}[a,z]$ For this purpose certain image formulas using Pathway fractional integral operators are obtained with these properties. Some important special cases of the main results are also mentioned.

**Key words:** Pathway Fractional Integral Operator, Generalized Mittag-Leffler Function, Generalized Function.
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## 1 Introduction:

**Pathway Integral Operator:** By using the Pathway idea of Mathai [5] and further defined by Mathai and Haubold [4], [6], a pathway fractional integral operator is introduced by Nair [9] in the following form.

Let $f(x)\in L(a,b), \delta\in C, R(\delta)>0, a>0$ and the pathway parameter $\alpha<1$, then the pathway fractional integral operator is defined as

$$\left(P_{0+}^{\delta,\alpha,a} f\right)(x) = x^{\delta}\int_{0}^{\frac{x}{a(1-\alpha)}}\left[1-\frac{a(1-\alpha)t}{x}\right]^{\frac{\delta}{1-\alpha}} f(t)dt \tag{1.1}$$

If we take $\alpha=0, a=1$, replace $\delta=\delta-1$ in (1.1) then we have the following relationship:

$$\left(P_{0+}^{(\delta-1,0,1)} f\right)(x) = \Gamma(\delta)\left[\left(I_{0+}^{\delta} f\right)(x)\right] \tag{1.2}$$

Where $I_{0+}^{\delta}$ is the left-sided Riemann-Liouville fractional integral operator [12].

Special G and R function:

The special $G_{\rho,\eta,\gamma}[a,z]$ is defined by [2], [3] as

$$G_{\rho,\eta,\gamma}[a,z] = z^{\gamma\rho-\eta-1}\sum_{n=0}^{\infty}\frac{(\gamma)_n (az^{\rho})^n}{\Gamma(n\rho+\rho\gamma-\eta)n!} \tag{1.3}$$

At $\gamma=1$, and $z$ replaced by $(z-c)$ it reduces to the special R function

$$R_{\rho,\eta}[a,c,z] = (z-c)^{\rho-\eta-1}\sum_{n=0}^{\infty}\frac{[a(z-c)^{\rho}]^n}{\Gamma(n\rho+\rho-\eta)n!}, \rho \ge 0, \rho \ge \eta. \tag{1.4}$$

**Generalized Mittag-Leffler Function:** A generalization of Mittag-Leffler function is introduced and defined by Prabhakar [11] as

$$E_{\lambda,\beta}^{\gamma}[z] = \sum_{n=0}^{\infty}\frac{(\gamma)_n}{\Gamma(\lambda n+\beta)n!} z^n \tag{1.5}$$

Where $\lambda,\beta,\gamma \in C\,(R(\lambda)>0)$.

## 2 Some properties of function $G_{\rho,\eta,\gamma}[a,z]$:

For $\rho,\eta,\gamma,\omega,\sigma,q,\alpha \in C, (Re(\rho), Re(\eta), Re(q), Re(\alpha) > 0)$ and $n \in N$ there hold the following properties for the special function $G_{\rho,\eta,\gamma}[a,z]$ [7]

**Property-1**

$$\left(\frac{d}{dz}\right)^n [G_{\rho,\eta,\gamma}[\omega,z]] = G_{\rho,\eta+n,\gamma}[\omega,z] \tag{2.1}$$

**Property-2**

$$\int_0^x G_{\rho,\eta,\gamma}[\omega,(x-t)]G_{\rho,\eta,\sigma}[\omega,t]dt = G_{\rho,\eta+q,\gamma+\sigma}[\omega,x] \tag{2.2}$$

**Property-3**

$$I_{a+}^{\alpha}(G_{\rho,\eta,\gamma}[\omega,(t-a)])(x) = G_{\rho,\eta-\alpha,\gamma}[\omega,(x-a)], (x>a) \tag{2.3}$$

**Property-4**

$$D_{a+}^{\alpha}(G_{\rho,\eta,\gamma}[\omega,(t-a)])(x) = G_{\rho,\eta+\alpha,\gamma}[\omega,(x-a)], (x>a) \tag{2.4}$$

## 3 Results Required :

The following result is required here [4, pg. 239], [8](also see [1])

$$P_{o+}^{(\delta,\alpha,c)}\left\{t^{\beta-1}\right\}=\frac{t^{\delta+\beta}}{[c(1-\alpha)]^{\beta}}\frac{\Gamma(\beta)\Gamma\left(1+\frac{\delta}{1-\alpha}\right)}{\Gamma\left(\frac{\delta}{1-\alpha}+\beta+1\right)} \tag{3.1}$$

where $\alpha<1; Re(\delta)>0; Re(\beta)>0$.

## 4 Pathway Fractional Integration:

**Theorem 4.1** Let $\alpha<1, \rho,\eta,\gamma,\omega,\sigma,q\in C$, then for

$$Re(\delta)>0; Re(\beta)>0, Re(\rho), Re(\eta), Re(q), Re(\alpha)>0$$

$$P_{o+}^{(\delta,\alpha,c)}\left\{\left(\frac{d}{dz}\right)^{n}\left[G_{\rho,\eta,\gamma}[\omega,z]\right]\right\}=\mathrm{C}\,G_{\rho,\eta+n-\frac{\delta}{1-\alpha},\gamma}\left[\omega,\left(\frac{z}{c(1-\alpha)}\right)^{\rho}\right]$$

Where $\mathrm{C}=\dfrac{\Gamma\left(1+\frac{\delta}{1-\alpha}\right)}{\left(\frac{z}{[c(1-\alpha)]^{\delta}}\right)^{\frac{1}{1-\alpha}}}$

**Proof:** Using the property-1 (2.1) on left hand side of (4.1) and applying (1.3), we reach at

$$P_{o+}^{(\delta,\alpha,c)}\left\{\left(\frac{d}{dz}\right)^{n}\left[G_{\rho,\eta,\gamma}[\omega,z]\right]\right\}=P_{o+}^{(\delta,\alpha,c)}\left\{z^{\gamma\rho-\eta-n-1}\sum_{n=0}^{\infty}\frac{(\gamma)_n\left(\omega z^{\rho}\right)^{n}}{\Gamma(\rho\gamma+\rho n-\eta-n)n!}\right\}$$

$$=\sum_{n=0}^{\infty}\frac{(\gamma)_n(\omega)^n}{\Gamma(\rho\gamma+\rho n-\eta-n)n!}P_{0+}^{\delta,\alpha,c}\left\{z^{\rho\gamma-\eta-n+\rho n-1}\right\}$$

Now using the formula of Pathway fractional integral operator (3.1)

$$=\sum_{n=0}^{\infty}\frac{(\gamma)_n\omega^n\quad\Gamma\left(1+\frac{\delta}{1-\alpha}\right)\quad z^{\delta+\rho\gamma+\rho n-\eta-n}}{[c(1-\alpha)]^{\rho\gamma+\rho n-\eta-n}\Gamma\left(\rho\gamma+\rho n-\eta-n+1+\frac{\delta}{1-\alpha}\right)}$$

$$=z^{\delta}\left[\frac{z}{[c(1-\alpha)]}\right]^{\rho\gamma-\eta-n+1-1}\Gamma\left(1+\frac{\delta}{1-\alpha}\right)$$

$$\times\sum_{n=0}^{\infty}\frac{(\gamma)_n\omega^n}{\Gamma\left(\rho\gamma+\rho n-\eta-n+1+\frac{\delta}{1-\alpha}\right)}\left[\frac{z}{c(1-\alpha)}\right]^{\rho n}$$

On using (1.3) we get the desired result of (4.1)

**Theorem 4.2** Let $\rho,\eta,\gamma,\omega,\sigma,q,\alpha \in C$, then for

$Re(\delta)>0; Re(\beta)>0,\ Re(\rho), Re(\eta), Re(q), Re(\alpha)>0, t>0$

$$P_{0+}^{\delta,\gamma,a}\left[\int_0^x G_{\rho,\eta,\gamma}[\omega,(x-t)]G_{\rho,q,\sigma}[\omega,t]dt\right]$$

$$= C\, G_{\rho,p-\frac{\delta}{1-\alpha}-1,r}\left[\omega,\frac{x}{[a(1-\alpha)]}\right]$$

Where $C = \dfrac{\Gamma\left(1+\dfrac{\delta}{1-\alpha}\right)}{\left[\dfrac{x}{[a(1-\alpha)]^{\delta}}\right]^{\frac{1}{1-\alpha}}}$

**Proof:** Using the property-2(2.2) on the left hand side of (4.2) and applying (1.3) we reach at

$$P_{0+}^{\delta,\gamma,a}\left[\int_0^x G_{\rho,\eta,\gamma}[\omega,(x-t)]G_{\rho,q,\sigma}[\omega,t]dt\right]$$

$$= P_{0+}^{\delta,\gamma,a}\left[x^{\rho(\gamma+\sigma)-\eta-q-1}\sum_{n=0}^{\infty}\frac{(\gamma+\sigma)_n(\omega x^{\rho})^n}{\Gamma(\rho n+\rho(\gamma+\sigma)-\eta-q)n!}\right]$$

$$= \sum_{n=0}^{\infty}\frac{(\gamma+\sigma)_n\omega^n}{\Gamma(\rho n+\rho(\gamma+\sigma)-(\eta+q))n!}P_{0+}^{\delta,\gamma,a}\left\{x^{\rho(\gamma+\sigma+n)-\eta-q-1}\right\}$$

Using the Pathway fractional integral operator (3.1)

$$= \sum_{n=0}^{\infty}\frac{(\gamma+\sigma)_n\omega^n\Gamma\left(1+\frac{\delta}{1-\alpha}\right)}{n![a(1-\alpha)]^{\rho(\gamma+\sigma+n)-(\eta+q)}}\frac{x^{\delta+\rho(\gamma+\sigma)+\rho n-(\eta+q)}}{\Gamma\left(\rho(\gamma+\sigma)+\rho n+\frac{\delta}{1-\alpha}-(\eta+q)+1\right)}$$

$$= \frac{x^{\delta+\rho r-p}\Gamma\left(1+\frac{\delta}{1-\alpha}\right)}{[a(1-\alpha)]^{\rho r-p}}\sum_{n=0}^{\infty}\frac{(r)_n\omega^n x^{\rho n}}{[a(1-\alpha)]^{\rho n}\Gamma\left(\rho r+\rho n+\frac{\delta}{1-\alpha}-p+1\right)}$$

$$= \frac{\left[\frac{x}{a(1-\alpha)}\right]^{\frac{\delta}{1-\alpha}+r\rho-p+1-1}\Gamma\left(1+\frac{\delta}{1-\alpha}\right)}{\left[\frac{x}{[a(1-\alpha)]^{\delta}}\right]^{\frac{1}{1-\alpha}}}\sum_{n=0}^{\infty}\frac{(r)_n\left(\omega\left(\frac{x}{a(1-\alpha)}\right)^{\rho}\right)^n}{\Gamma\left(\rho r+\rho n+\frac{\delta}{1-\alpha}-p+1\right)n!}$$

On using (1.3) we reach at the required result of (4.2).

**Theorem 4.3** Let $\rho,\eta,\gamma,\omega,\alpha \in C$, then for

$Re(\delta)>0; Re(\zeta)>0,\ Re(\rho), Re(\eta), Re(q), Re(\alpha)>0, t>0, x>0$

$$P_{0+}^{\delta,\zeta,c}\left\{I_{a+}^{\alpha}G_{\rho,\eta,\gamma}[\omega,(t-a)]\right\}(x)=\mathrm{C}\,G_{\rho,p,r}\left(\omega,\frac{x-a}{c(1-\zeta)}\right)$$

$$\mathrm{C}=\frac{\Gamma\left(1+\frac{\delta}{1-\zeta}\right)}{\left[\frac{x-a}{[c(1-\zeta)]^{\delta}}\right]^{\frac{1}{1-\zeta}}}\quad and \quad p=\eta-\alpha-\frac{\delta}{1-\zeta}-1$$

**Proof:** Using Property-3 (2.3) on the left hand side of (4.3) and applying (1.3) we get

$$P_{0+}^{\delta,\zeta,c}\left\{I_{a+}^{\alpha}G_{\rho,\eta,\gamma}[\omega,(t-a)]\right\}(x)=P_{0+}^{\delta,\zeta,c}(x-a)^{\rho\gamma-(\eta-\alpha)-1}\sum_{n=0}^{\infty}\frac{(\gamma)_n\left(\omega(x-a)^{\rho}\right)^n}{\Gamma(\rho n+\rho\gamma-(\eta-\alpha))n!}$$

$$=\sum_{n=0}^{\infty}\frac{(\gamma)_n\omega^n}{\Gamma(\rho n+\rho\gamma-(\eta-\alpha))n!}P_{0+}^{\delta,\zeta,c}(x-a)^{\rho\gamma+\rho n-(\eta-\alpha)-1}$$

$$=\sum_{n=0}^{\infty}\frac{(\gamma)_n\omega^n\Gamma\left(1+\frac{\delta}{1-\zeta}\right)}{\Gamma\left(\rho n+\rho\gamma-(\eta-\alpha)+\frac{\delta}{1-\zeta}+1\right)}(x-a)^{\delta}\left(\frac{x-a}{c(1-\zeta)}\right)^{\rho\gamma+\rho n-(\eta-\alpha)}$$

$$=(x-a)^{\delta}\left[\frac{x-a}{c(1-\zeta)}\right]^{\rho\gamma-(\eta-\alpha)+1-1}\Gamma\left(1+\frac{\delta}{1-\zeta}\right)\times\sum_{n=0}^{\infty}\frac{(\gamma)_n\omega^n\left[\frac{x-a}{c(1-\alpha)}\right]^{\rho n}}{\Gamma\left(\rho n+\rho\gamma-(\eta-\alpha)+\frac{\delta}{1-\zeta}+1\right)}$$

$$=\frac{(x-a)^{\delta}\left[\frac{x-a}{[c(1-\zeta)]^{\delta}}\right]^{\frac{1}{1-\zeta}+\rho\gamma-(\eta-\alpha)+1-1}}{\left[\frac{x-a}{[c(1-\zeta)]^{\delta}}\right]^{\frac{1}{1-\zeta}}}\Gamma\left(1+\frac{\delta}{1-\zeta}\right)\times\sum_{n=0}^{\infty}\frac{(\gamma)_n\omega^n\left[\frac{x-a}{c(1-\alpha)}\right]^{\rho n}}{\Gamma\left(\rho n+\rho\gamma-\left(\eta-\alpha-\frac{\delta}{1-\zeta}-1\right)\right)}$$

Taking $\eta-\alpha-\frac{\delta}{1-\zeta}-1=p$ and using (1.3) we reach at the desired result of (4.3)

**Corollary 4.4** If we put $\eta=\rho\gamma-\mu$ in theorem 4.3, then we get the following result

$$P_{0+}^{\delta,\zeta,c}\left\{I_{a+}^{\alpha}G_{\rho,\rho\gamma-\mu,\gamma}\left[\omega,(t-a)\right]\right\}(x)$$

$$=\frac{(x-a)^{\delta+\mu+\alpha}\Gamma\left(1+\frac{\delta}{1-\zeta}\right)}{\left[c(1-\zeta)\right]^{\mu+\alpha}\Gamma(\gamma)}\,{}_1\psi_1\left[\begin{matrix}(\gamma,1)\\ \left(1+\mu+\alpha+\frac{\delta}{1-\zeta},\rho\right)\end{matrix};\omega\left(\frac{x-a}{c(1-\zeta)}\right)^{\rho}\right]$$

**Theorem 4.5** *Let* $\rho,\eta,\gamma,\omega,\alpha\in C$ , then for
$Re(\delta)>0; Re(\zeta)>0,\ Re(\rho), Re(\eta), Re(q), Re(\alpha)>0, t>0, x>0$

$$P_{0+}^{\delta,\zeta,c}\left\{D_{a+}^{\alpha}G_{\rho,\eta,\gamma}\left[\omega,(t-a)\right]\right\}(x)=\mathsf{C}\,G_{\rho,\eta+\alpha-\frac{\delta}{1-\zeta}-1,\gamma}\left[\omega,\left(\frac{x-a}{c(1-\zeta)}\right)^{\rho}\right]$$

Where $\mathsf{C}=\dfrac{\Gamma\left(1+\frac{\delta}{1-\zeta}\right)}{\left(\frac{x-a}{\left[c(1-\zeta)\right]^{\delta}}\right)^{\frac{1}{1-\zeta}}}$

**Proof:** Using property -4 (2.4) on the left hand side of (4.5) and applying (1.3) we reach at

$$P_{0+}^{\delta,\zeta,c}\left\{D_{a+}^{\alpha}G_{\rho,\eta,\gamma}\left[\omega,(t-a)\right]\right\}(x)=P_{0+}^{\delta,\zeta,c}\left[(x-a)^{\rho\gamma-\eta-\alpha-1}\sum_{n=0}^{\infty}\frac{(\gamma)_n\left[\omega(x-a)^{\rho}\right]^n}{\Gamma(\rho n+\rho\gamma-\eta-\alpha)n!}\right]$$

$$=\sum_{n=0}^{\infty}\frac{(\gamma)_n\omega^n}{\Gamma(\rho n+\rho\gamma-\eta-\alpha)n!}P_{0+}^{\delta,\zeta,c}\left\{(x-a)^{\rho\gamma+\rho n-\eta-\alpha-1}\right\}$$

$$=\sum_{n=0}^{\infty}\frac{(\gamma)_n\omega^n}{n!}\frac{(x-a)^{\delta+\rho\gamma+\rho n-\eta-\alpha}}{\left[c(1-\zeta)\right]^{\rho\gamma+\rho n-\eta-\alpha}}\frac{\Gamma\left(1+\frac{\delta}{1-\zeta}\right)}{\Gamma\left(\rho\gamma+\rho n-\eta-\alpha+1+\frac{\delta}{1-\zeta}\right)}$$

$$=\frac{(x-a)^{\delta+\rho\gamma-\eta-\alpha}\Gamma\left(1+\frac{\delta}{1-\zeta}\right)}{\left[c(1-\zeta)\right]^{\rho\gamma-\eta-\alpha}}\sum_{n=0}^{\infty}\frac{(\gamma)_n\omega^n}{\Gamma\left(\rho\gamma+\rho n-\eta-\alpha+1+\frac{\delta}{1-\zeta}\right)n!}\left[\frac{(x-a)}{\left[c(1-\zeta)\right]}\right]^{\rho n}$$

On using the definition (1.3) we directly reach to result of (4.5).

## 5 Special Cases:

1. If we set $\eta = \rho\gamma - \mu$ then the above theorems 4.1 and 4.5 reduces to the well known image formula for generalized Mittag-Leffler function under the Pathway integral operator, of [9] eq.(24) also see [10].
2. If we set $\eta = \rho\gamma - \mu$ and $q = \rho\sigma - \nu$, then the functional of left hand side of 4.2 and 4.3 reduces to generalized Mittag-Leffler function and the theorems leads to the well-known image formula of [9].

## 6 Concluding Remarks:

The present paper deals with the Pathway Fractional Integration of generalized function $G_{\rho,\eta,r}[a,z]$ using its properties. Additionally the derived theorems also lead to Pathway fractional integration of type of generalized Mittag-Leffler function as special cases. This paper is concluded with the remark that the reported results are significant and can be used to yield number of other image formulas.

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