Contra Harmonic index of Operations on Graphs

[1]**S.Ragavi,**
Assistant Professor, PG and Research Department of Mathematics, Mannar Thirumalai Naicker College, Madurai.
stragavi22@gmail.com
[2]**R.Sridevi,**
Assistant Professor, PG and Research Department of Mathematics,
Sri S.R.N.M. College, Sattur.
sridevi@srnmcollege.ac.in

Abstract

Let G be a simple graph with vertex set V(G) and edge set E(G) respectively. The degree d(v) denotes the degree of the vertex in graph G. Topological index is a quantitative parameter which describes the topology of structural framework. The advantage of topological indices may be used as numerical descriptors in chemical parameters of molecules in QSPR and QSAR. Contra Harmonic index CH(G) of a simple graph G is defined as the sum of the terms $\frac{d(u)^2+d(v)^2}{d(u)+d(v)}$ over all edges uv of G, where d(u) and d(v) denotes the degree of vertices u and v in G respectively.

$$CH(G) = \sum_{uv\in E(G)} \frac{d(u)^2 + d(v)^2}{d(u) + d(v)}$$

In this paper, we compute the contra harmonic index of operation on graphs such as Join, Corono product and Cartesian product of two graphs.

Keywords: Contra Harmonic Index, Join, Corono product, Cartesian Product

AMS Classification: 05C76, 05C09

Introduction:

The first and second Zagreb indices $M_1(G)$ and $M_2(G)$ were introduced 30 years ago by Gutman and Trinajstic[5]. They are defined as

$$M_1(G) = \sum_{v\in V(G)} d(v)^2$$

$$M_2(G) = \sum_{uv\in E(G)} d(u)d(v)$$

Fortula and Gutman [2] introduced Forgotten topological index which was defined as
$$F(G) = \sum_{v \epsilon V(G)} d(v)^3 = \sum_{uv \epsilon E(G)} d(u)^2 + d(v)^2$$

The inverse sum degree [10] of a connected graph G is defined as $ID(G) = \sum_{u \in V(G)} \frac{1}{d(u)}$

The symmetric division deg index [3] of a connected graph G is defined as
$$SDD(G) = \sum_{uv \in E(G)} \frac{d(u)^2 + d(v)^2}{d(u)d(v)}$$

In [1],some sharp bounds for the inverse sum indeg index of connected graphs are given. K. Pattabiraman presented several upper and lower bounds on the inverse sum indeg index in terms of some molecular structural parameters [7] and [8].

Contra Harmonic index CH(G) of a simple graph G is defined as the sum of the terms $\frac{d(u)^2+d(v)^2}{d(u)+d(v)}$ over all edges uv of G, where d(u) and d(v) denotes the degree of vertices u and v in G respectively[9].
$$CH(G) = \sum_{uv \in E(G)} \frac{d(u)^2 + d(v)^2}{d(u) + d(v)}$$

Theorem 1

Let G_1 and G_2 be two graphs on n_1, n_2 vertices and m_1, m_2 edges respectively. Then
$CH(G_1 + G_2) \geq \frac{1}{2\Delta_1+2n_2}\left[F(G_1) + 2n_2M_1(G_1) + 2{n_2}^2m_1\right] + \frac{1}{2\Delta_2+2n_1}\left[F(G_2) + 2n_1M_1(G_2) + 2{n_1}^2m_2\right] + \frac{1}{\Delta_1+\Delta_2+n_1+n_2}\left[n_2M_1(G_1) + 4{n_2}^2m_1 + n_1M_1(G_2) + ({n_2}^2 + {n_1}^2)n_1n_2 + 4{n_1}^2m_2\right]$

Proof:

Let $v_1, v_2, \dots, v_{n_1}$ be the vertices of G_1 and $v_1, v_2, \dots, v_{n_2}$ be the vertices of G_2 respectively.

By the definition of $G_1 + G_2$ we have
$$d_{G_1+G_2}(u) = \begin{cases} d_{G_1}(u) + n_2 & if\ u \in V(G_1) \\ d_{G_2}(u) + n_1 & if\ u \in V(G_2) \end{cases}$$

$$CH(G_1 + G_2) = \sum_{uv \in E(G_1+G_2)} \frac{{d^2}_{G_1+G_2}(u) + {d^2}_{G_1+G_2}(v)}{d_{G_1+G_2}(u) + d_{G_1+G_2}(v)}$$
$$= \sum_{uv \in E(G_1)} \frac{(d_{G_1}(u) + n_2)^2 + (d_{G_1}(v) + n_2)^2}{d_{G_1}(u) + n_2 + d_{G_1}(v) + n_2} + \sum_{uv \in E(G_2)} \frac{(d_{G_2}(u) + n_1)^2 + (d_{G_2}(v) + n_1)^2}{d_{G_2}(u) + n_1 + d_{G_2}(v) + n_1}$$
$$+ \sum_{u \in V(G_1), v \in V(G_2)} \frac{(d_{G_1}(u) + n_2)^2 + (d_{G_1}(v) + n_1)^2}{d_{G_1}(u) + n_2 + d_{G_1}(v) + n_1}$$
$$= I_1 + I_2 + I_3$$

Take,
$$I_1 = \sum_{uv \in E(G_1)} \frac{{d^2}_{G_1}(u) + 2d_{G_1}(u)n_2 + {n_2}^2 + {d^2}_{G_1}(v) + 2d_{G_1}(v)n_2 + {n_2}^2}{d_{G_1}(u) + d_{G_1}(v) + 2n_2}$$

Since, $\delta \leq d(v) \leq \Delta$, we have

$$I_1 \geq \frac{1}{2\Delta_1 + 2n_2}\left[\sum_{uv\in E(G_1)}\left[d^2{}_{G_1}(u) + d^2{}_{G_1}(v)\right] + 2n_2\sum_{uv\in E(G_1)}\left[d_{G_1}(u) + d_{G_1}(v)\right] + 2{n_2}^2\sum_{uv\in E(G_1)} 1\right]$$

$$= \frac{1}{2\Delta_2 + 2n_2}\left[F(G_1) + 2n_2M_1(G_1) + 2{n_2}^2m_1\right]$$

Similarly, $I_2 \geq \frac{1}{2\Delta_2+2n_1}\left[F(G_2) + 2n_1M_1(G_2) + 2{n_1}^2m_2\right]$

$$I_3 = \sum_{u\in V(G_1)}\sum_{v\in V(G_2)} \frac{(d_{G_1}(u) + n_2)^2 + (d_{G_2}(v) + n_1)^2}{d_{G_1}(u) + n_2 + d_{G_2}(v) + n_1}$$

$$\geq \frac{1}{\Delta_1 + \Delta_2 + n_1 + n_2}\sum_{u\in V(G_1)}\sum_{v\in E(G_2)}\left[d^2{}_{G_1}(u) + 2d_{G_1}(u)n_2 + {n_2}^2 + d^2{}_{G_2}(v) + 2d_{G_2}(v)n_1 + {n_1}^2\right]$$

$$= \frac{1}{\Delta_1 + \Delta_2 + n_1 + n_2}\left[\sum_{u\in V(G_1)}\sum_{v\in V(G_2)} d^2{}_{G_1}(u) + 2n_2\sum_{u\in V(G_1)}\sum_{v\in V(G_2)} d_{G_1}(u) + \sum_{u\in V(G_1)}\sum_{v\in V(G_2)} d^2{}_{G_2}(v) + ({n_2}^2 + {n_1}^2)\sum_{u\in V(G_1)}\sum_{v\in V(G_2)} 1 + 2n_1\sum_{u\in V(G_1)}\sum_{v\in V(G_2)} d_{G_2}(v)\right]$$

$$\geq \frac{1}{2\Delta_1 + 2n_2}\left[\sum_{uv\in E(G_1)}\left[d^2{}_{G_1}(u) + d^2{}_{G_1}(v)\right] + 2n_2\sum_{uv\in E(G_1)}\left[d_{G_1}(u) + d_{G_1}(v)\right] + 2{n_2}^2\sum_{uv\in E(G_1)} 1\right]$$

$$= \frac{1}{2\Delta_2 + 2n_2}\left[F(G_1) + 2n_2M_1(G_1) + 2{n_2}^2m_1\right]$$

Since, $\sum d(v) = 2m,$ we have

$$I_3 \geq \frac{1}{\Delta_1 + \Delta_2 + n_1 + n_2}\left[n_2\sum_{u\in V(G_1)} d^2{}_{G_1}(u) + 2{n_2}^2\sum_{u\in V(G_1)} d_{G_1}(u) + \sum_{u\in V(G_1)} M_1(G_2) + ({n_2}^2 + {n_1}^2)n_1n_2 + 2n_1(2m_2)\sum_{u\in V(G_1)} 1\right]$$

$$I_3 \geq \frac{1}{\Delta_1 + \Delta_2 + n_1 + n_2}\left[n_2M_1(G_1) + 4{n_2}^2m_1 + n_1M_1(G_2) + ({n_2}^2 + {n_1}^2)n_1n_2 + 4{n_1}^2m_2\right]$$

Substitute the value of I_1, I_2 and I_3 in I, we get

$CH(G_1 + G_2) \geq \frac{1}{2\Delta_1+2n_2}\left[F(G_1) + 2n_2M_1(G_1) + 2{n_2}^2m_1\right] + \frac{1}{2\Delta_2+2n_1}\left[F(G_2) + 2n_1M_1(G_2) + 2{n_1}^2m_2\right] + \frac{1}{\Delta_1+\Delta_2+n_1+n_2}\left[n_2M_1(G_1) + 4{n_2}^2m_1 + n_1M_1(G_2) + ({n_2}^2 + {n_1}^2)n_1n_2 + 4{n_1}^2m_2\right]$

Theorem 2

Let G_1 and G_2 be two graphs on n_1, n_2 vertices and m_1, m_2 edges respectively. Then $CH(G_1\,o\,G_2) \geq \frac{1}{2\Delta_1+2n_2}\left[F(G_1) + 2n_2M_1(G_1) + 2{n_2}^2m_1\right] + \frac{1}{2\Delta_2+2n_1}\left[F(G_2) + 2M_1(G_2) + 2m_2\right] + \frac{1}{\Delta_1+\Delta_2+n_1+n_2}\left[n_2M_1(G_1) + 4{n_2}^2m_1 + n_1M_1(G_2) + ({n_2}^2 + 1)n_1n_2 + 4{n_1}^2m_2\right]$

Proof:

Let $v_1, v_2, \dots, v_{n_1}$ be the vertices of G_1 and $v_1, v_2, \dots, v_{n_2}$ be the vertices of G_2 respectively.
The edge set of $G_1\,o\,G_2$ can be partitioned int three subsets E_1, E_2 and E_3 as follows:

$E_1 = \{e \in E(G_1\,o\,G_2) | e \in E(G_1)\}$

$E_2 = \{e \in E(G_1\,o\,G_2) | e \in E(G_{2,1})\}$ and

$E_3 = \{e \in E(G_1\,o\,G_2) | e - uv, u \in V(G_1)\ and\ v \in V(G_{2,i})\}$ where $G_{2,i}$ is the i^{th} copy of G_2 and $i = 1,2,\dots,n_1$.

By the definition of $G_1\,o\,G_2$ we have

$$d_{G_1\,o\,G_2}(u) = \begin{cases} d_{G_1}(u) + n_2 & if\ u \in V(G_1) \\ d_{G_2}(u) + 1 & if\ u \in V(G_2) \end{cases}$$

$$CH(G_1\,o\,G_2) = \sum_{uv \in E(G_1\,o\,G_2)} \frac{{d^2}_{G_1\,o\,G_2}(u) + {d^2}_{G_1\,o\,G_2}(v)}{d_{G_1\,o\,G_2}(u) + d_{G_1\,o\,G_2}(v)}$$

$$= \sum_{uv \in E(G_1)} \frac{{d^2}_{G_1\,o\,G_2}(u) + {d^2}_{G_1\,o\,G_2}(v)}{d_{G_1\,o\,G_2}(u) + d_{G_1\,o\,G_2}(v)} + \sum_{uv \in E(G_2)} \frac{{d^2}_{G_1\,o\,G_2}(u) + {d^2}_{G_1\,o\,G_2}(v)}{d_{G_1\,o\,G_2}(u) + d_{G_1\,o\,G_2}(v)}$$

$$+ \sum_{u \in V(G_1) v \in V(G_2)} \frac{{d^2}_{G_1\,o\,G_2}(u) + {d^2}_{G_1\,o\,G_2}(v)}{d_{G_1\,o\,G_2}(u) + d_{G_1\,o\,G_2}(v)}$$

$$= I_1 + I_2 + I_3$$

$$I_1 = \sum_{uv \in E(G_1)} \frac{\left(d_{G_1}(u) + n_2\right)^2 + \left(d_{G_1}(v) + n_2\right)^2}{d_{G_1}(u) + n_2 + d_{G_1}(v) + n_2}$$

$$= \sum_{uv \in E(G_1)} \frac{{d^2}_{G_1}(u) + 2d_{G_1}(u)n_2 + {n_2}^2 + {d^2}_{G_1}(v) + 2d_{G_1}(v)n_2 + {n_2}^2}{d_{G_1}(u) + d_{G_1}(v) + 2n_2}$$

$$\geq \frac{1}{2\Delta_1 + 2n_2}\left[\sum_{uv \in E(G_1)} \left[{d^2}_{G_1}(u) + {d^2}_{G_1}(v)\right] + 2n_2 \sum_{uv \in E(G_1)} \left[d_{G_1}(u) + d_{G_1}(v)\right] + 2{n_2}^2 \sum_{uv \in E(G_1)} 1\right]$$

$$= \frac{1}{2\Delta_2 + 2n_2}\left[F(G_1) + 2n_2M_1(G_1) + 2{n_2}^2m_1\right]$$

$$I_2 = \sum_{uv\in E(G_2)} \frac{(d_{G_2}(u)+1)^2 + (d_{G_2}(v)+1)^2}{d_{G_2}(u)+1+d_{G_2}(v)+1}$$
$$= \sum_{uv\in E(G_2)} \frac{d^2{}_{G_2}(u) + 2d_{G_2}(u) + 1 + d^2{}_{G_2}(v) + 2d_{G_2}(v) + 1}{d_{G_2}(u)+d_{G_2}(v)+2}$$
$$\geq \frac{1}{2\Delta_1+2}\left[\sum_{uv\in E(G_2)}\left[d^2{}_{G_2}(u)+d^2{}_{G_2}(v)\right] + 2\sum_{uv\in E(G_2)}\left[d_{G_2}(u)+d_{G_2}(v)\right] + 2\sum_{uv\in E(G_2)} 1\right]$$
$$= \frac{1}{2\Delta_2+2n_1}[F(G_2)+2M_1(G_2)+2m_2]$$

$$I_3 = \sum_{u\in V(G_1)}\sum_{v\in VG_2)} \frac{(d_{G_1}(u)+n_2)^2 + (d_{G_2}(v)+1)^2}{d_{G_1}(u)+n_2+d_{G_2}(v)+1}$$
$$\geq \frac{1}{\Delta_1+\Delta_2+n_1+n_2}\sum_{u\in V(G_1)}\sum_{v\in V(G_2)}\left[d^2{}_{G_1}(u) + 2d_{G_1}(u)n_2 + n_2{}^2 + d^2{}_{G_2}(v) + 2d_{G_2}(v) + 1\right]$$
$$= \frac{1}{\Delta_1+\Delta_2+n_1+n_2}\left[\sum_{u\in V(G_1)}\sum_{v\in V(G_2)} d^2{}_{G_1}(u) + 2n_2\sum_{u\in V(G_1)}\sum_{v\in V(G_2)} d_{G_1}(u) + \sum_{u\in V(G_1)}\sum_{v\in V(G_2)} d^2{}_{G_2}(v) + (n_2{}^2+1)\sum_{u\in V(G_1)}\sum_{v\in V(G_2)} 1 + 2\sum_{u\in V(G_1)}\sum_{v\in V(G_2)} d_{G_2}(v)\right]$$
$$= \frac{1}{\Delta_1+\Delta_2+n_1+n_2}\left[n_2\sum_{u\in V(G_1)} d^2{}_{G_1}(u) + 2n_2{}^2\sum_{u\in V(G_1)} d_{G_1}(u) + \sum_{u\in V(G_1)} M_1(G_2) + (n_2{}^2+1)n_1n_2 + 2n_1(2m_2)\sum_{u\in V(G_1)} 1\right]$$
$$= \frac{1}{\Delta_1+\Delta_2+n_1+n_2}[n_2M_1(G_1) + 4n_2{}^2m_1 + n_1M_1(G_2) + (n_2{}^2+1)n_1n_2 + 4n_1{}^2m_2]$$

Substitute the value of I_1, I_2 and I_3 in I, we get

$$CH(G_1 o G_2) \geq \frac{1}{2\Delta_1+2n_2}\left[F(G_1) + 2n_2M_1(G_1) + 2n_2{}^2m_1\right] + \frac{1}{2\Delta_2+2n_1}[F(G_2) + 2M_1(G_2) + 2m_2] + \frac{1}{\Delta_1+\Delta_2+n_1+n_2}[n_2M_1(G_1) + 4n_2{}^2m_1 + n_1M_1(G_2) + (n_2{}^2+1)n_1n_2 + 4n_1{}^2m_2]$$

Lemma

For any vertices x, y, z of a graph G and for any positive integers n_1, n_2 and n_3

$$\frac{1}{\frac{1}{n_1+n_2+n_3}(n_1d_G(x)+n_2d_G(y)+n_3d_G(z))} \leq \frac{1}{n_1+n_2+n_3}\left[\frac{n_1}{d_G(x)} + \frac{n_2}{d_G(y)} + \frac{1}{d_G(z)}\right]$$

Theorem 3

Let G_1 and G_2 be two graphs on n_1, n_2 vertices and m_1, m_2 edges respectively. Then
$CH(G_1 \times G_2) \leq \frac{1}{16}[32m_1m_2 + 6[n_2M_1(G_1) + n_1M_1(G_2)] + 2[F(G_2)ID(G_1) + F(G_2)ID(G_1)] + [n_1F(G_2) + n_2F(G_1)] + 4[m_1SDD(G_2) + m_2SDD(G_1)]]$

Proof:

Let $u_1, u_2, \dots, u_{n_1}$ be the vertices of G_1 and $v_1, v_2, \dots, v_{n_2}$ be the vertices of G_2 respectively.
By the definition of $G_1 \times G_2$, the number of vertices and edges of $G_1 \times G_2$ is n_1n_2 and $m_1n_2 + m_2n_1$ respectively. The degree of a vertex (u_i, v_j) is $d_{G_1}(u_i) + d_{G_2}(v_j)$.

$$CH(G_1 \times G_2) = \sum_{(u_i,v_j)(u_k,v_l)\in E(G_1\times G_2)} \frac{d^2{}_{G_1\times G_2}\left((u_i,v_j)\right) + d^2{}_{G_1\times G_2}\left((u_k,v_l)\right)}{d_{G_1\times G_2}\left((u_i,v_j)\right) + d_{G_1\times G_2}\left((u_k,v_l)\right)}$$

$$= \sum_{(u_i,v_j)(u_k,v_l)\in E(G_1\times G_2), v_jv_j\in E(G_2)} \frac{d^2{}_{G_1\times G_2}\left((u_i,v_j)\right) + d^2{}_{G_1\times G_2}\left((u_k,v_l)\right)}{d_{G_1\times G_2}\left((u_i,v_j)\right) + d_{G_1\times G_2}\left((u_k,v_l)\right)}$$

$$+ \sum_{(u_i,v_j)(u_k,v_l)\in E(G_1\times G_2), u_iu_k\in E(G_1)} \frac{d^2{}_{G_1\times G_2}\left((u_i,v_j)\right) + d^2{}_{G_1\times G_2}\left((u_k,v_l)\right)}{d_{G_1\times G_2}\left((u_i,v_j)\right) + d_{G_1\times G_2}\left((u_k,v_l)\right)}$$

$$= I_1 + I_2$$

$$I_1 = \sum_{(u_i,v_j)(u_k,v_l)\in E(G_1\times G_2), v_jv_j\in E(G_2)} \frac{d^2{}_{G_1\times G_2}\left((u_i,v_j)\right) + d^2{}_{G_1\times G_2}\left((u_k,v_l)\right)}{d_{G_1\times G_2}\left((u_i,v_j)\right) + d_{G_1\times G_2}\left((u_k,v_l)\right)}$$

$$= \sum_{u_i\in V(G_1)} \sum_{v_jv_j\in E(G_2)} \frac{\left(d_{G_1}(u_i) + d_{G_2}(v_j)\right)^2 + \left(d_{G_1}(u_i) + d_{G_2}(v_l)\right)^2}{d_{G_1}(u_i) + d_{G_2}(v_j) + d_{G_1}(u_i) + d_{G_2}(v_l)}$$

$$= \sum_{u_i\in V(G_1)} \sum_{v_jv_j\in E(G_2)} \frac{d^2{}_{G_1}(u_i) + d^2{}_{G_2}(v_j) + 2d_{G_1}(u_i)d_{G_2}(v_j) + d^2{}_{G_1}(u_i) + d^2{}_{G_2}(v_l) + 2d_{G_1}(u_i)d_{G_2}(v_l)}{2d_{G_1}(u_i) + d_{G_2}(v_j) + d_{G_2}(v_l)}$$

$$= \sum_{u_i\in V(G_1)} \sum_{v_jv_j\in E(G_2)} \frac{2d^2{}_{G_1}(u_i)}{2d_{G_1}(u_i) + d_{G_2}(v_j) + d_{G_2}(v_l)}$$

$$+ \sum_{u_i\in V(G_1)} \sum_{v_jv_j\in E(G_2)} \frac{d^2{}_{G_2}(v_j) + d^2{}_{G_2}(v_l)}{2d_{G_1}(u_i) + d_{G_2}(v_j) + d_{G_2}(v_l)}$$

$$+ \sum_{u_i\in V(G_1)} \sum_{v_jv_j\in E(G_2)} \frac{2d_{G_1}(u_i)[d_{G_2}(v_j) + d_{G_2}(v_l)]}{2d_{G_1}(u_i) + d_{G_2}(v_j) + d_{G_2}(v_l)}$$

$$= I_1' + I_1'' + I_1'''$$

$$I_1' = \sum_{u_i\in V(G_1)} \sum_{v_jv_j\in E(G_2)} \frac{2d^2{}_{G_1}(u_i)}{2d_{G_1}(u_i) + d_{G_2}(v_j) + d_{G_2}(v_l)}$$

$$\leq \frac{2}{16} \sum_{u_i\in V(G_1)} \sum_{v_jv_j\in E(G_2)} d^2{}_{G_1}(u_i) \left[\frac{2}{d_{G_1}(u_i)} + \frac{1}{d_{G_2}(v_j)} + \frac{1}{d_{G_2}(v_l)}\right]$$

$$= \frac{1}{8}\left[\sum_{u_i\in V(G_1)}\sum_{v_jv_j\in E(G_2)} 2d_{G_1}(u_i) + \sum_{u_i\in V(G_1)}\sum_{v_jv_j\in E(G_2)} d^2{}_{G_1}(u_i)\left[\frac{1}{d_{G_2}(v_j)} + \frac{1}{d_{G_2}(v_l)}\right]\right]$$

$$= \frac{1}{8}\left[\sum_{u_i\in V(G_1)} 2m_2 d_{G_1}(u_i) + \sum_{u_i\in V(G_1)} d^2{}_{G_1}(u_i)n_2\right]$$

$$= \frac{1}{8}[2m_2(2m_1) + n_2M_1(G_1)]$$

$$I_1' \le \frac{1}{8}[4m_1m_2 + n_2M_1(G_1)]$$

$$I_1'' = \sum_{u_i\in V(G_1)}\sum_{v_jv_l\in E(G_2)} \frac{d^2{}_{G_2}(v_j) + d^2{}_{G_2}(v_l)}{2d_{G_1}(u_i) + d_{G_2}(v_j) + d_{G_2}(v_l)}$$

$$\le \frac{1}{16}\sum_{u_i\in V(G_1)}\sum_{v_jv_l\in E(G_2)} [d^2{}_{G_2}(v_j) + d^2{}_{G_2}(v_l)]\left[\frac{2}{d_{G_1}(u_i)} + \frac{1}{d_{G_2}(v_j)} + \frac{1}{d_{G_2}(v_l)}\right]$$

$$= \frac{1}{16}\left[\sum_{u_i\in V(G_1)}\sum_{v_jv_l\in E(G_2)} \frac{2[d^2{}_{G_2}(v_j) + d^2{}_{G_2}(v_l)]}{d_{G_1}(u_i)} + \sum_{u_i\in V(G_1)}\sum_{v_jv_l\in E(G_2)} [d^2{}_{G_2}(v_j) + d^2{}_{G_2}(v_l)]\left[\frac{1}{d_{G_2}(v_j)} + \frac{1}{d_{G_2}(v_l)}\right]\right]$$

$$= \frac{1}{16}\left[\sum_{u_i\in V(G_1)} \frac{2}{d_{G_1}(u_i)}F(G_2) + \sum_{u_i\in V(G_1)} F(G_2)\right]$$

$$= \frac{1}{16}[2F(G_2)ID(G_1) + n_1F(G_2)]$$

$$I_1''' = \sum_{u_i\in V(G_1)}\sum_{v_jv_l\in E(G_2)} \frac{2d_{G_1}(u_i)[d_{G_2}(v_j) + d_{G_2}(v_l)]}{2d_{G_1}(u_i) + d_{G_2}(v_j) + d_{G_2}(v_l)}$$

$$\le \frac{2}{16}\sum_{u_i\in V(G_1)}\sum_{v_jv_j\in E(G_2)} d_{G_1}(u_i)[d_{G_2}(v_j) + d_{G_2}(v_l)]\left[\frac{2}{d_{G_1}(u_i)} + \frac{1}{d_{G_2}(v_j)} + \frac{1}{d_{G_2}(v_l)}\right]$$

$$= \frac{1}{8}\left[\sum_{u_i\in V(G_1)}\sum_{v_jv_j\in E(G_2)} 2[d_{G_2}(v_j) + d_{G_2}(v_l)] + \sum_{u_i\in V(G_1)}\sum_{v_jv_j\in E(G_2)} d_{G_1}(u_i)[d_{G_2}(v_j) + d_{G_2}(v_l)]\left[\frac{1}{d_{G_2}(v_j)} + \frac{1}{d_{G_2}(v_l)}\right]\right]$$

$$= \frac{1}{8}\left[\sum_{u_i\in V(G_1)} 2M_1(G_2) + \sum_{u_i\in V(G_1)}\sum_{v_jv_j\in E(G_2)} d_{G_1}(u_i)\frac{[d_{G_2}(v_j) + d_{G_2}(v_l)]^2}{d_{G_2}(v_j)d_{G_2}(v_l)}\right]$$

$$= \frac{1}{8}\left[\sum_{u_i \in V(G_1)} 2M_1(G_2) + \sum_{u_i \in V(G_1)} \sum_{v_j v_j \in E(G_2)} \left[d_{G_1}(u_i)\frac{d^2{}_{G_2}(v_j) + d^2{}_{G_2}(v_l)}{d_{G_2}(v_j)d_{G_2}(v_l)} + d_{G_1}(u_i)\frac{2d_{G_2}(v_j)d_{G_2}(v_l)}{d_{G_2}(v_j)d_{G_2}(v_l)}\right]\right]$$

$$= \frac{1}{8}\left[2n_1M_1(G_2) + \sum_{u_i \in V(G_1)} \left[d_{G_1}(u_i)SDD(G_2) + 2d_{G_1}(u_i)m_2\right]\right]$$

$$= \frac{1}{8}[2n_1M_1(G_2) + 2m_1SDD(G_2) + 4m_1m_2]$$

Substitute the values of I_1', I_1'' and I_1''' in I_1, we get

$$I_1 \leq \frac{1}{16}[16m_1m_2 + 2n_2M_1(G_1) + 2F(G_2)ID(G_1) + n_1F(G_2) + 4n_1M_1(G_2) + 4m_1SDD(G_2)]$$

Similarly,

$$I_2 \leq \frac{1}{16}[16m_1m_2 + 2n_1M_1(G_2) + 2F(G_1)ID(G_2) + n_2F(G_1) + 4n_2M_1(G_1) + 4m_2SDD(G_1)]$$

$$CH(G_1 \times G_2) \leq \frac{1}{16}[32m_1m_2 + 6[n_2M_1(G_1) + n_1M_1(G_2)] + 2[F(G_2)ID(G_1) + F(G_2)ID(G_1)] + [n_1F(G_2) + n_2F(G_1)] + 4[m_1SDD(G_2) + m_2SDD(G_1)]]$$

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