A Refinement of an $L_{\mathtt{q}}$ Inequality for a Polynomial with Restrictions on Zeros

Aijaz Ahmad[1], Mushtaq Ahmad Shah[1], Prabal Pratap Singh[2]

[1]Department of Mathematics
GCET, Safapora, Ganderbal, Jammu and Kashmir, India, 193501
e-mail: aiju.aijaz117@gmail.com, mushtaqa022@gmail.com

[2]Department of Mathematics, J.S. University,
Shikohabad(Firozabad), U.P, India.
e-mail: drppsingh0508@gmail.com

Abstract

Let F_η be the set of all polynomials of degree η, then for $F \in F_\eta$, the $L_{\mathtt{q}}$ analogue of Bernstein's inequality was proved by Zygmund. In fact, he proved that

$$\|F'\|_{\mathtt{q}} \leq \eta \|F\|_{\mathtt{q}}, \quad for \;\; \mathtt{q} > 0.$$

In literature, so many generalizations and refinements of this result exists. Recently K Krishnadas and B Chanam [10,Theorem 1] proved a generalisation of above result. In this paper we prove a refinement of above result which in turn provides a generalization of several other results.

Mathematics Subject Classification: 26D10, 30C15, 41A17.
Keywords and Phrases: Polynomials, integral mean inequalities, Zeros, $L_{\mathtt{q}}$ norm.

1 Introduction

Let F_η be the set of all polynomials of degree η.

For $F \in F_\eta$ define,

$$\|F\|_{\mathtt{q}} := \left\{ \frac{1}{2\pi} \int_0^{2\pi} |F(e^{i\Theta})|^{\mathtt{q}} d\Theta \right\}^{\frac{1}{\mathtt{q}}}, \quad 0 < \mathtt{q} < \infty$$

$$\|F\|_\infty := \max_{|z|=1} |F(z)|$$

Also,

$$\|F\|_0 := exp \left\{ \frac{1}{2\pi} \int_0^{2\pi} log|F(e^{i\Theta}|d\Theta \right\}$$

If $F \in F_\eta$,

$$\|F'\|_{\mathtt{q}} \le \eta \|F\|_{\mathtt{q}}, \quad \mathtt{q} > 0. \tag{1.1}$$

inequality (1.1) was found out by Zygmund [18]. Letting $\mathtt{q} \to \infty$ in (1.1), we get

$$\|F'\|_\infty \le \eta \|F\|_\infty \tag{1.2}$$

which is a well-known Bernstein's inequality.
Arestov[1] proved that (1.1) is true for $\mathtt{q} \in [0,1)$ as well.
If $F \in F_\eta$ such that F(z) doesnot vanish in $|z| < 1$, then (1.2) and (1.1) can be respectively replaced by

$$\|F'\|_\infty \le \frac{\eta}{2} \|F\|_\infty \tag{1.3}$$

and

$$\|F'\|_{\mathtt{q}} \le \frac{\eta}{\|1+z\|_{\mathtt{q}}} \|F\|_{\mathtt{q}}, \quad \mathtt{q} > 0. \tag{1.4}$$

The inequality (1.3) was conjectured by Erdös and later verified by Lax [10], inequality (1.4) is due to Debruijn [4] for $\mathtt{q} \ge 1$. Rahman and Schmeisser[14] verified that (1.4) is valid for $\mathtt{q} \in [0,1)$ as well.
Turan[18] proved that, if $F \in F_\eta$ vanishes in $|z| \le 1$,then

$$\|F'\|_\infty \ge \frac{\eta}{2} \|F\|_\infty. \tag{1.5}$$

equality holds in both the inequalities (1.3) and (1.5) for $F(z) = l + \tau z^{\eta}$, where $|l| = |\tau|$.
Also, Govil and Rahman [7] proved that, if $F \in F_{\eta}$ does not vanishes in $|z| < h, h \geq 1$, then

$$\|F'\|_{\mathtt{q}} \leq \frac{\eta}{\|h + z\|_{\mathtt{q}}} \|F\|_{\mathtt{q}}, \quad \mathtt{q} \geq 1 \tag{1.6}$$

Gardener and Weems [6] and Rather [14] independently verified the validity of (1.6) for $0 < \mathtt{q} < 1$ as well.
Also, Aziz and Rather[2] proved that if $F \in F_{\eta}$ does not vanishes in $|z| < h, h \geq 1$, then for every $\mathtt{q} > 0$

$$\|F'\|_{\mathtt{q}} \leq \frac{\eta}{\|\Delta_{h,1} + z\|_{\mathtt{q}}} \|F\|_{\mathtt{q}}, \quad \mathtt{q} \geq 1 \tag{1.7}$$

where $\Delta_{h,1} = \dfrac{\eta|a_0|h^2 + h^2|a_1|}{\eta|a_0| + h^2|a_1|}$

Malik[12] proved that if $F \in F_{\eta}$ vanishes in $|z| \leq 1$, then for every $\mathtt{q} positive.$

$$\|F'\|_{\mathtt{q}} \geq \frac{\eta}{\|1 + z\|_{\mathtt{q}}} \|F\|_{\mathtt{q}}. \tag{1.8}$$

Aziz and Rather [2] generalised the inequality (1.8) as if $F \in F_{\eta}$ vanishes in $|z| \leq h, h \leq 1$, then for every $\mathtt{q} > 0$

$$\|F'\|_{\mathtt{q}} \geq \frac{\eta}{\|1 + t_{h,1} z\|_{\mathtt{q}}} \|F\|_{\mathtt{q}}. \tag{1.9}$$

and

$$\|F'\|_{\infty} \geq \frac{\eta}{\|1 + t_{h,1} z\|_{\mathtt{q}}} \|F\|_{\mathtt{q}}. \tag{1.10}$$

where $t_{h,1} = \dfrac{\eta|a_{\eta}|h^2 + |a_{\eta-1}|}{\eta|a_{\eta}| + |a_{\eta-1}|}$

Govil et al. [8] demonstrated the following two theorems, in which they generalised (1.6) and (1.8), and also the inequality (1.11) by involving some coefficients of F(z).

Theorem 1.1. If $F \in F_\eta$ does not vanishes in $|z| < h, h \geq 1$, then

$$\|F'\|_\infty \leq \frac{\eta}{1+h} \frac{(1-|\alpha|)(1+h^2|\alpha|) + h(\eta-1)|\mu-\alpha^2|}{(1-|\alpha|)(1-h+h^2+h|\alpha|) + h(\eta-1)|\mu-\alpha^2|} \|F\|_\infty, \tag{1.11}$$

where

$$\alpha = \frac{ha_1}{\eta a_0} \text{ and } \mu = \frac{2h^2 a_2}{\eta(\eta-1)a_0}.$$

Theorem 1.2. If $F \in F_\eta$ vanishes in $|z| \leq h, h \leq 1$, then

$$\|F'\|_\infty \geq \frac{\eta}{1+h} \frac{(1-|\beta|)(1+h^2|\beta|) + h(\eta-1)|\gamma-\beta^2|}{(1-|\beta|)(1-h+h^2+h|\beta|) + h(\eta-1)|\gamma-\beta^2|} \|F\|_\infty, \tag{1.12}$$

where

$$\beta = \frac{\overline{a}_{\eta-1}}{\eta h \overline{a}_\eta} \text{ and } \gamma = \frac{2\overline{a}_{\eta-2}}{\eta(\eta-1)h^2\overline{a}_\eta}.$$

Recently, Krishnadas and Chanam[10] demonstrated the following two theorems in which they extended inequalities (1.14) and (1.15) to L_{q} norms.
,

Theorem 1.3. If $F \in F_\eta$ does not vanishes in $|z| < h, h \geq 1$, then for every $\mathsf{q} > 0$

$$\|F'\|_{\mathsf{q}} \leq \frac{\eta}{\|\Gamma + z\|_{\mathsf{q}}} \|F\|_{\mathsf{q}}. \tag{1.13}$$

where

$$\Gamma = h \frac{(1-|\alpha|)(|\alpha|+h^2) + h(\eta-1)|\mu-\alpha^2|}{(1-|\alpha|)(|\alpha|h^2+1) + h(\eta-1)|\mu-\alpha^2|} \tag{1.14}$$

$$\alpha = \frac{ha_1}{\eta a_0} \text{ and } \mu = \frac{2h^2 a_2}{\eta(\eta-1)a_0}.$$

Theorem 1.4. If $F \in F_\eta$ vanishes in $|z| \leq h, h \leq 1$, then for every $\mathsf{q} > 0$

$$\|F'\|_{\mathsf{q}} \geq \frac{\eta}{\|1 + Sz\|_{\mathsf{q}}} \|F\|_{\mathsf{q}}. \tag{1.15}$$

where

$$S = h\frac{(1-|\beta|)(|\beta|+h^2)+h(\eta-1)|\gamma-\beta^2|}{(1-|\beta|)(|\beta|h^2+1)+h(\eta-1)|\gamma-\beta^2|} \tag{1.16}$$

$$\beta = \frac{\overline{a}_{\eta-1}}{\eta h\overline{a}_\eta} \text{ and } \gamma = \frac{2\overline{a}_{\eta-2}}{\eta(\eta-1)h^2\overline{a}_\eta}.$$

2 Main Results

In this paper,first we obtain the following result which includes not only a refinement of Theorem 1.3 but also provides some generalization of other results.

Theorem 2.1. If $F \in F_\eta$ does not vanishes in $|z| < h, h \geq 1$, then for every $\mathtt{q} > 0$

$$\|F' + \tau\eta m z^{\eta-1}\|_{\mathtt{q}} \leq \frac{\eta}{\|\Gamma + z\|_{\mathtt{q}}}\|F\|_{\mathtt{q}}. \tag{2.1}$$

where Γ is defined by (1.17) and $m = \underset{|z|=h}{min}|F(z)|$.

Remark 2.2 If we put $m = 0$ and let $\mathtt{q} \to \infty$ in (2.1), the Theorem 2.1 reduces to Theorem 1.1 by using the inequality (1.17).

Remark 2.3 For $m = 0$, inequality (2.1) reduces to inequality (1.16). Next, we prove the theorem as a refinement of Theorem 1.4. In fact we prove

Theorem 2.4. If $F \in F_\eta$ vanishes in $|z| \leq h, h \leq 1$, then for every $\mathtt{q} > 0$

$$\|F' + \tau\eta m z^{\eta-1}\|_{\mathtt{q}} \geq \frac{\eta}{\|1 + Sz\|_{\mathtt{q}}}\|F\|_{\mathtt{q}}. \tag{2.2}$$

where S is defined by (1.19)

Remark 2.5. For $m = 0$ Theorem 2.4 reduces to Theorem 1.4.

Instead of proving Theorem 2.4, we prove a more general result, from which Theorem 2.4,follows as a special case.

Theorem 2.6. If $F \in F_\eta$ vanishes in $|z| \leq h, h \leq 1$, then for every $\mathtt{q} > 0, r > 1, s > 1$ with $r^{-1} + s^{-1} = 1$.

$$\|F' + \tau\eta m z^{\eta-1}\|_{s\mathtt{q}} \geq \frac{\eta}{\|1 + Sz\|_{r\mathtt{q}}}\|F\|_{\mathtt{q}}. \tag{2.3}$$

where S is defined by (1.19).

Remark 2.7. Put $r = 0$ or $s = 0$, we obtain Theorem 2.4.

3 Lemmas

In this section we present some lemmas which will help us to prove our results.

Lemma 3.1. If $F \in F_\eta$ does not vanishes in $|z| < h, h \geq 1$, then

$$\Gamma|F'(z)| \leq |\mathcal{Q}'(z)| \tag{3.1}$$

where Γ is defined by (1.17) and $\mathcal{Q}(z) = z^\eta \overline{F(\frac{1}{\bar{z}})}$

Above Lemma 3.1 is due to Govil et al.[8]. By applying Lemma (3.1.) to the polynomial $F(z) = F(z) + m\tau z^\eta$, we immediately get the following result.

Lemma 3.2. If $F \in F_\eta$ does not vanishes in $|z| < h, h \geq 1$, then for any complex number τ with $|\tau| \leq 1$,

$$\Gamma|F'(z) + \tau\eta m z^{n-1}| \leq |\mathcal{Q}'(z)| \tag{3.2}$$

where Γ is defined by (1.17) and $m = \underset{|z|=h}{min}|F(z)|$.

Lemma 3.3. If $F \in F_\eta$ vanishes in $|z| \leq h, h \leq 1$, then on $|z| = 1$

$$|\mathcal{Q}'(z)| \leq S|F'(z) + \eta m \tau z^{\eta-1}| \tag{3.3}$$

where S is defined by (1.19).

Proof of Lemma 3.3. Since $F(z)$ vanishes in $|z| \leq h, h \leq 1$, then the polynomial $\mathcal{Q}(z) = z^\eta \overline{F(\frac{1}{\bar{z}})}$ does not vanishes in $|z| < \frac{1}{h}, \frac{1}{h} \geq 1$. Thus applying Lemma 3.2 to the polynomial $\mathcal{Q}(z)$, we have

$$|\mathcal{Q}'(z)| \leq h\frac{(1-|\beta|)(|\frac{1}{h^2}\beta|+1)+\frac{1}{h}(\eta-1)|\gamma-\beta^2|}{(1-|\beta|)(|\beta|+h^2)+\frac{1}{h}(\eta-1)|\gamma-\beta^2|}\left|F'(z)+\eta m\tau z^{\eta-1}\right|$$

$\beta = \frac{1/h}{\eta}\frac{\overline{a}_{\eta-1}}{\overline{a}_{\eta}} = \frac{\overline{a}_{\eta-1}}{\eta h\overline{a}_{\eta}}$ and $\gamma = \frac{2/h^2}{\eta(\eta-1)h^2}\frac{\overline{a}_{\eta-2}}{\overline{a}_{\eta}} = \frac{2\overline{a}_{\eta-2}}{\eta(\eta-1)h^2\overline{a}_{\eta}}$.
Then,

$$|\mathcal{Q}'(z)| \leq h\frac{(1-|\beta|)(|\beta|+h^2)+h(\eta-1)|\gamma-\beta^2|}{(1-|\beta|)(|\beta|h^2+1)+h(\eta-1)|\gamma-\beta^2|}\left|F'(z)+\eta m\tau z^{\eta-1}\right|$$

which proves Lemma 3.3.

Lemma 3.4. If $F \in F_\eta$, then for every l, $0 \leq l < 2\pi$ and $\mathtt{q} > 0$

$$\int_0^{2\pi}\int_0^{2\pi}|\mathcal{Q}'(e^{i\Theta})+e^{il}F'(e^{i\Theta})|^{\mathtt{q}}d\Theta dl \leq 2\pi\eta^{\mathtt{q}}\int_0^{2\pi}|F(e^{i\Theta})|^{\mathtt{q}}d\Theta \tag{3.4}$$

The above lemma is due to Aziz [2]

Lemma 3.5. If $F \in F_\eta$, then for every l, $0 \leq l < 2\pi$,$\mathtt{q} > 0$ and for any complex number τ with $|\tau| \leq 1$,

$$\int_0^{2\pi}\int_0^{2\pi}|\mathcal{Q}'(e^{i\Theta})+e^{il}\left\{F'(e^{i\Theta})+\eta m\tau e^{i(\eta-1)\Theta}\right\}|^{\mathtt{q}}d\Theta dl \leq 2\pi\eta^{\mathtt{q}}\int_0^{2\pi}|F(e^{i\Theta})|^{\mathtt{q}}d\Theta \tag{3.5}$$

Proof of Lemma 3.5. By applying Lemma (3.4.) to the polynomial $F(z) = F(z) + m\tau z^n$, we can easily get the proof of Lemma 3.5.

Lemma 3.6. Let z be any complex and independent of l , where l is any real, then for $\mathtt{q} > 0$

$$\int_0^{2\pi}|1+ze^{il}|^{\mathtt{q}}dl = \int_0^{2\pi}|e^{il}+|z||^{\mathtt{q}}dl \tag{3.6}$$

Lemma 3.6 is due to Gardner and Govil[5].

4 Proofs of Theorems

Proof of Theorem 2.1. As $F(z)$ does not vanishes in $|z| < h, h \geq 1$ hence, by Lemma 3.2. we have

$$\Gamma|F'(z) + \tau\eta m z^{\eta-1}| \leq |\mathcal{Q}'(z)| \tag{4.1}$$

where Γ is defined by (1.17) and $m = \underset{|z|=h}{min}|F(z)|$.
For each Real l and $G \geq r \geq 1$, we have

$$|G + e^{il}| \geq |r + e^{il}|$$

Then, for every $\mathtt{q} > 0$, we have

$$\int_0^{2\pi} |G + e^{il}|^{\mathtt{q}} dl \geq \int_0^{2\pi} |r + e^{il}|^{\mathtt{q}} dl \tag{4.2}$$

For points $e^{i\Theta}$, $0 \leq \Theta < 2\pi$, for which $F'(e^{i\Theta}) + \eta m\tau e^{i(\eta-1)\Theta}$ does not vanishes, we denote $G = \left|\frac{\mathcal{Q}'(e^{i\Theta})}{F'(e^{i\Theta}) + \eta m\tau e^{i(\eta-1)\Theta}}\right|$ and $r = \Gamma$, then by (4.1), we for every $\mathtt{q} > 0$

$$\begin{aligned}
&\int_0^{2\pi} \left|\mathcal{Q}'(e^{i\Theta}) + e^{il}\left\{F'(e^{i\Theta}) + \eta m\tau e^{i(\eta-1)\Theta}\right\}\right|^{\mathtt{q}} dl \\
&= \left|F'(e^{i\Theta}) + \eta m\tau e^{i(\eta-1)\Theta}\right|^{\mathtt{q}} \int_0^{2\pi} \left|\frac{\mathcal{Q}'(e^{i\Theta})}{F'(e^{i\Theta}) + \eta m\tau e^{i(\eta-1)\Theta}} + e^{il}\right|^{\mathtt{q}} dl \\
&= \left|F'(e^{i\Theta}) + \eta m\tau e^{i(\eta-1)\Theta}\right|^{\mathtt{q}} \int_0^{2\pi} \left|\left|\frac{\mathcal{Q}'(e^{i\Theta})}{F'(e^{i\Theta}) + \eta m\tau e^{i(\eta-1)\Theta}}\right| + e^{il}\right|^{\mathtt{q}} dl \qquad by \quad (3.6) \\
&= \left|F'(e^{i\Theta}) + \eta m\tau e^{i(\eta-1)\Theta}\right|^{\mathtt{q}} \int_0^{2\pi} \left|G + e^{il}\right|^{\mathtt{q}} dl \\
&\geq \left|F'(e^{i\Theta}) + \eta m\tau e^{i(\eta-1)\Theta}\right|^{\mathtt{q}} \int_0^{2\pi} \left|r + e^{il}\right|^{\mathtt{q}} dl \qquad by \quad (4.2)
\end{aligned}$$

Hence,

$$\int_0^{2\pi} \left|\mathcal{Q}'(e^{i\Theta}) + e^{il}\left\{F'(e^{i\Theta}) + \eta m\tau e^{i(\eta-1)\Theta}\right\}\right|^{\mathtt{q}} dl \geq \left|F'(e^{i\Theta}) + \eta m\tau e^{i(\eta-1)\Theta}\right|^{\mathtt{q}} \int_0^{2\pi} \left|\Gamma + e^{il}\right|^{\mathtt{q}} dl \tag{4.3}$$

for $e^{i\Theta}$, $0 \leq \Theta < 2\pi$, for which $F'(e^{i\Theta}) + \eta m\tau e^{i(\eta-1)\Theta}$ does not vanishes. For points $e^{i\Theta}$, $0 \leq \Theta < 2\pi$, for which $F'(e^{i\Theta}) + nm\tau e^{i(n-1)\Theta}$ vanishes, (4.3) trivially holds. Hence, using (4.3) in Lemma 3.5, we get for each $\mathtt{q} > 0$,

$$\int_0^{2\pi} \left|F'(e^{i\Theta}) + \eta m\tau e^{i(\eta-1)\Theta}\right|^{\mathtt{q}} d\Theta \int_0^{2\pi} \left|\Gamma + e^{il}\right|^{\mathtt{q}} \leq 2\pi\eta^{\mathtt{q}} \int_0^{2\pi} |F(e^{i\Theta})|^{\mathtt{q}} d\Theta$$

which is equivalent to

$$\left\{\int_0^{2\pi} \left|F'(e^{i\Theta}) + \eta m\tau e^{i(\eta-1)\Theta}\right|^{\mathtt{q}} d\Theta\right\}^{\frac{1}{\mathtt{q}}} \leq \eta \left\{\frac{1}{2\pi}\int_0^{2\pi} \left|\Gamma + e^{il}\right|^{\mathtt{q}} dl\right\}^{-\frac{1}{\mathtt{q}}} \left\{\int_0^{2\pi} \left|F(e^{i\Theta})\right|^{\mathtt{q}} d\Theta\right\}^{\frac{1}{\mathtt{q}}} \tag{4.4}$$

which proves Theorem 2.1.

Proof of Theorem 2.6. Since $F(z)$ vanishes in $|z| \leq h, h \leq 1$, $F(z)$ also vanishes in $|z| \leq h, h \leq 1$. Hence, by Gauss-Lucas Theorem,

$$z^{\eta-1}\overline{F(\frac{1}{\overline{z}})} = \eta\mathcal{Q}(z) - z\mathcal{Q}'(z) \tag{4.5}$$

vanishes in $|z| \geq \frac{1}{h}, \frac{1}{h} \geq 1$. Further, since $F(z)$ vanishes in $|z| \leq h, h \leq 1$, we have by Lemma 3.3.

$$\begin{aligned} |\mathcal{Q}'(z)| &\leq S|F'(z) + \eta m\tau z^{\eta-1}| \\ &= S\left\{|F'(z)| + t\eta m\right\} \quad for \ \ |z| = 1. \end{aligned} \tag{4.6}$$

where S is defined by (1.19) and $|\tau| = t$.
For $|z| = 1$, we also have

$$|F'(z) = |\eta\mathcal{Q}(z) - z\mathcal{Q}'(z)|. \tag{4.7}$$

Using (4.7) in (4.6) , we have on $|z| = 1$

$$|\mathcal{Q}'(z)| \leq S\left\{|\eta\mathcal{Q}(z) - z\mathcal{Q}'(z)| + t\eta m\right\} \tag{4.8}$$

Thus, by (4.5) and (4.8),

$$\psi(z) = \frac{z\mathcal{Q}'(z)}{S\left\{|\eta\mathcal{Q}(z) - z\mathcal{Q}'(z)| + t\eta m\right\}}$$

is analytic in $|z| \leq 1, |\psi(z)| \leq 1$ on $|z| = 1$ and $\psi(0) = 0$. Therefore, $1 + S\psi(z)$ is subordinate to the function $1 + Sz$ for $|z| \leq 1$. Hence, by a well known property of subordination [9], we have for every $\mathtt{q} > 0$

$$\int_0^{2\pi} |1 + S\psi(e^{i\Theta})|^{\mathtt{q}} d\Theta \geq \int_0^{2\pi} |1 + Se^{i\Theta}|^{\mathtt{q}} d\Theta \tag{4.9}$$

Now,

$$1 + S\psi(z) = 1 + \frac{z\mathcal{Q}'(z)}{|\eta\mathcal{Q}(z) - z\mathcal{Q}'(z)| + t\eta m} = \frac{\eta\mathcal{Q}(z)}{|\eta\mathcal{Q}(z) - z\mathcal{Q}'(z)| + t\eta m}$$

which implies for $|z| = 1$,

$$\begin{aligned} |\eta\mathcal{Q}(z)| =& |1 + S\psi(z)| \, | \, |\eta\mathcal{Q}(z) - z\mathcal{Q}'(z)| + t\eta m \\ &= |1 + S\psi(z)| \, | \, |F'(z)| + t\eta m \quad by \quad (4.7) \end{aligned}$$

As $|F(z)| = |\mathcal{Q}(z)|$ on $|z| = 1$, by preceding inequality we have

$$\eta|F(z)| = |1 + S\psi(z)| \, | \, |F'(z)| + t\eta m \quad on \quad |z| = 1 \tag{4.10}$$

Then for every $\mathtt{q} > 0$ and $0 \leq \Theta < 2\pi$, we have

$$\eta^{\mathtt{q}} \int_0^{2\pi} |F(e^{i\Theta})|^{\mathtt{q}} d\Theta = \int_0^{2\pi} |1 + S\psi(e^{i\Theta})|^{\mathtt{q}} \left\{ \left| F'(e^{i\Theta}) \right| + t\eta m \right\}^{\mathtt{q}} d\Theta$$

Applying Holder's inequality to the above inequality, we have for $r > 1, s > 1$ with $r^{-1} + s^{-1} = 1$ and for every $\mathtt{q} > 0$

$$\eta^{\mathtt{q}} \int_0^{2\pi} |F(e^{i\Theta})|^{\mathtt{q}} d\Theta \leq \left\{ \int_0^{2\pi} \left| 1 + S\psi(e^{i\Theta}) \right|^{r\mathtt{q}} d\Theta \right\}^{\frac{1}{r}} \left\{ \int_0^{2\pi} \left\{ \left| F'(e^{i\Theta}) \right| + t\eta m \right\}^{s\mathtt{q}} d\Theta \right\}^{\frac{1}{s}}$$

which implies

$$\eta \left\{ \int_0^{2\pi} |F(e^{i\Theta})|^{\mathtt{q}} d\Theta \right\}^{\frac{1}{\mathtt{q}}} \leq \left\{ \int_0^{2\pi} \left| 1 + S\psi(e^{i\Theta}) \right|^{r\mathtt{q}} d\Theta \right\}^{\frac{1}{\mathtt{q}r}} \left\{ \int_0^{2\pi} \left\{ \left| F'(e^{i\Theta}) \right| + t\eta m \right\}^{s\mathtt{q}} d\Theta \right\}^{\frac{1}{s\mathtt{q}}}$$

using (4.9) in the above inequality , we have

$$\eta\left\{\int_0^{2\pi}\left|F(e^{i\Theta})\right|^{\mathrm{q}}d\Theta\right\}^{\frac{1}{\mathrm{q}}}\leq\left\{\int_0^{2\pi}\left|1+Se^{i\Theta}\right|^{r\mathrm{q}}d\Theta\right\}^{\frac{1}{\mathrm{q}r}}\left\{\int_0^{2\pi}\left\{\left|F'(e^{i\Theta})\right|+t\eta m\right\}^{s\mathrm{q}}d\Theta\right\}^{\frac{1}{s\mathrm{q}}}$$

by choosing argument of τ as in the proof of Theorem 2.4 , we get the above inequality as

$$\eta\left\{\int_0^{2\pi}\left|F(e^{i\Theta})\right|^{\mathrm{q}}d\Theta\right\}^{\frac{1}{\mathrm{q}}}\leq\left\{\int_0^{2\pi}\left|1+Se^{i\Theta}\right|^{r\mathrm{q}}d\Theta\right\}^{\frac{1}{\mathrm{q}r}}\left\{\int_0^{2\pi}\left|F'(e^{i\Theta})+\tau\eta me^{i(n-1)\Theta}\right|^{s\mathrm{q}}d\Theta\right\}^{\frac{1}{s\mathrm{q}}}$$

which proves the theorem.

References

[1] V. V. Arestov, On inequalities for trigonometric polynomials and their derivative, IZV. Akad. Nauk. SSSR. Ser. Math., 45 (1981), 3-22.

[2] . Aziz and N.A. Rather, Some Zygmund type Lq inequalities for polynomials, J. Math. Anal. Appl., 289 (2004), 14-29.

[3] . Bernstein, Lecons sur les proprietes extremales et la meilleure approx- imation desfonctions analytiques dune variable reelle, Gauthier Villars, Paris (1926).

[4] .G. de Bruijn, Inequalities concerning polynomials in the complex domain, Nederl. Akad. Wetensch. Proc., 50 (1947), 1265-1272; Indag. Math., 9 (1947), 591-598.

[5] .B. Gardner and N.K. Govil, An Lq inequality for a polynomial and its derivative, J. Math. Anal. Appl., 194 (1995), 720-726.

[6] . Gardner and A. Weems, A Bernstein-type Lp inequality for a certain class of polynomials, J. Math. Anal. Appl., 219 (1998), 472-478.

[7] .K. Govil and Q.I. Rahman, Functions of exponential type not vanishing in a half-plane and related polynomials, Trans. Amer. Math. Soc., 137 (1969), 501-517.

[8] .K. Govil, Q.I. Rahman and G. Schmeisser, On the derivative of a polynomial, Illinois J. of Math., 23, No 2 (1979), 319-329.

[9] . Hille, Analytic Function Theory, Vol. II, Ginn. and Company, New York, Toronto (1962).

[10] .Krishnadas and B Chanam, L^{γ} Inequalities Concerning Polynomials, International Journal of Applied Mathematics, Vol. 34, No. 5(2021) 979-994.

[11] .D. Lax, Proof of a Conjecture of P. Erdos on the derivative of a polynomial, Bull. Amer. Math. Soc., 50 (1944), 509-513.

[12] .A. Malik, On the derivative of a polynomial, J. London Math. Soc., (1969), 57-60.

[13] .A.Malik, An integral mean estimate for polynomials, Proc. Amer. Math. Soc., 91 (1984), 281-284.

[14] .I. Rahman and G. Schmeisser, Lp inequalities for polynomials, J. Approx. Theory, 53 (1998), 26-32.

[15] .A. Rather, Extremal Properties and Location of the Zeros of Polynomi- als, PhD. Thesis, University of Kashmir (19980.

[16] . Rudin, Real and Complex Analysis, Tata Mcgraw-Hill Publ. Co. (Reprinted in India) (1977).

[17] .E. Taylor, Introduction to Functional Analysis, John Wiley and Sons, Inc., New York (1958).

[18] . Turan, Uber die ableitung von polynomen, Compositio Math., 7 (1939), 89-95.

[19] . Zygmund, A remark on conjugate series, Proc. London Math. Soc., 34 (1932), 392-400.