# Fixed Point Theorems in T- Orbitally Complete Spaces with PGA-Quasi Contraction.

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## Abstract

Poom Kuman [6] has established the generalized version of the result by Ciric [3] By considering the most general form of quasi-contraction viz. PGA-Quasi contraction, we have established the existence of unique fixed point in T- orbitally complete spaces in this paper.


## 1 Introduction

Ciric generalized the Banach's contraction principle [3] by defining the quasi-contraction map in 1974 and proved Ciric Fixed Point theorem. In 2008, Berinde [2] defined cirictype almost contractions in metric spaces and established existence of fixed point. Lakshmikantham et al in 2009 [7] proved coupled fixed point theorem for nonlinear contraction considering partially ordered metric space. In this chapter, we have defined more general quasi contraction by adding the following factor G(x, y) to the generalized quasi contraction due to ciric [3] and established the existence and uniqueness of Fixed Point in T - Orbitally complete Metric Space.

## 2 Preliminaries

In order to prove the main result of this chapter, we need the following definitions and notions. Let (X, d) be the metric space and E, F be any two subsets of X then

$$D(E, F) = \inf\{d(a, b) : a \in E, b \in F\}$$

$$\rho(E, F) = \sup\{d(a, b) : a \in E, b \in F\}$$

$$\delta(E) = \sup\{d(a, b) : a, b \in E\}$$

**Definition 2.1.** [4]. Let $T : X \to X$ be a map on metric space (X, d). For each $x \in X$ and for any positive integer n, denote $O_T(x,n) = \{x, T x, \dots, T^n x\}$ and $O_T(x,+\infty) = \{x, T x, \dots, T^n x, \dots\}$. The set $O_T(x,+\infty)$ is called the orbit of T at x and the metric space X is called T- Orbitally complete if every Cauchy sequence in $O_T(x,+\infty)$ is convergent in X.

**Example 2.1.** Let (R, d) be metric space with respect to usual metric d, and $T : R \to R, T(x) = \frac{x}{4}$. Then orbit of T is $O_T(x,+\infty) = \{x, \frac{x}{4}, \dots, \frac{x}{4^n}, \dots\}$ and it may be verified easily that R is T- Orbitally complete.

**Definition 2.2.** [3] Let $T : X \to X$ be a mapping on metric space (X, d). The mapping

T is said to be a **quasi-contraction** if there exists q ∈ [0, 1) such that for all x, y ∈ X,

$$d(T\,x, T\,y) \le q.max\{d(x,y), d(x,T\,x), d(y,T\,y), d(y,T\,x), d(x,T\,y)\}$$

**Example 2.2.** Let (E, d) be metric space with respect to usual metric d, where $E =$

$[0,\infty)$ and $T: [0,\infty) \to [0,\infty), T(x) = \frac{x}{2}$.

It is clear that T satisfies quasi-contraction condition.

Referring the definition 2.2, we now introduce the generalized form of quasi-contraction due to ciric [3].

Definition 2.3. Let T : X → X be a mapping on metric space (X, d). The mapping T is said to be a PGA-quasi contraction if there exists q ∈ [0, 1) such that for all x, y ∈ X,

$$d(T\,x, T\,y) \le q.max\{d(x,y), d(x,T\,x), d(y,T\,y), d(y,T\,x), d(x,T\,y), G(x,y)\}.$$

Where

$$G(x,y) = \begin{cases} \frac{d(x,T\,x)d(x,T\,y)+d(y,T\,y)d(y,T\,x)}{\max\{d(x,T\,y),d(T\,x,y)\}} & if\ max\{d(x,T\,y), d(T\,x,y)\} \neq 0 \\ 0 & if\ max\{d(x,T\,y), d(T\,x,y)\} = 0 \end{cases}, \ \forall x,y \in X \quad (2.1)$$

Example 2.3. Let (R, d) be metric space with respect to usual metric d, and $: R \to R, T(x) = \frac{x}{8}$.

T is a PGA-quasi contraction map.

**3 Main Result** In this section, we state one of the two main results of this paper.

**Theorem 3.1.** Let (X, d) be the metric space and T : X → X be a PGA-quasi contraction map (cf. Definition 2.3). Also X is T- orbitally complete. Then T has a unique fixed point $x^*$ in X.

Proof. We first establish the existence of a fixed point under the map T.

For each x ∈ X and 1 ≤ i ≤ n − 1 and 1 ≤ j ≤ n, where n ∈ $Z_+$.

Consider

$$d(T^i x, T^j x) = d(T\,T^{i-1}x, T\,T^{j-1}x)$$

$$\le q.\max\{d(T^{i-1}x, T^{j-1}x), d(T^{i-1}x, T\,T^{i-1}x), d(T^{j-1}x, T\,T^{j-1}x),$$

$$d(T^{j-1}x, T\,T^{i-1}x), d(T^{i-1}x, T\,T^{j-1}x), G(T^{i-1}x, T^{j-1})\}$$

$$\le q.\max\{d(T^{i-1}x, T^{j-1}x), d(T^{i-1}x, T^i x), d(T^{j-1}x, T^j x),$$

$$d(T^{j-1}x, T^{i}x), d(T^{i-1}x, T^{j}x), d(T^{i+1}x, T^{i-1}x),$$

$$d(T^{i+1}x, T^{i}x), d(T^{i+1}x, T^{j-1}x), d(T^{i+1}x, T^{j}x), G(T^{i-1}x, T^{j-1}x)\}$$

$$\leq q.\delta[O_T(x,n)]$$

Where,

$\delta[O_T(x,n)] = \max\{d(T^{i}x, T^{j}x): 0 \leq i,j \leq n\}$. Since $0 \leq q < 1, \exists\, k_n(x) \leq n$ such that

$$d(x, T^{(k_n)(x)}x) = \delta[O_T(x,n)] \tag{3.1}$$

Now,

$$d(x, T^{k_n(x)}x) \leq d(x, Tx) + d(Tx, T^{k_{(n(x))}}x) \leq d(x, Tx) + q.[O_T(x,n)]$$
$$\leq d(x, Tx) + q.d(x, T^{k_n(x)}x)$$

It implies that

$$(1-q)d(x, T^{(k_n)(x)}x) \leq d(x, Tx)$$

$$d(x, T^{(k_n)(x)}x) \leq \frac{1}{(1-q)} d(x, Tx)$$

Using (3.1), we get $\delta[O_T(x,n)] = d(x, T^{(k_n)(x)}x) \leq \frac{1}{(1-q)} d(x, Tx)$ (3.2)

For all n, m ≤ 1 and n ≤ m, it follows from PGA-quasi contraction condition on T and (3.2) that

$$
\begin{aligned}
d(T^n x, T^m x) &= d(TT^{n-1}x, T^{m-n+1}T^{n-1}x) \\
&\leq q.\delta[O_T(T^{n-1}x, m-n+1)] \\
&\leq q.d(T^{n-1}x, T^{k_{m-n+1(T^{n-1}x)}}T^{n-1}x) \\
&\leq q.d(TT^{n-2}x, T^{k_{m-n+1(T^{n-1}x)}+1}T^{n-2}x) \\
&\leq q.d(TT^{n-2}x, T^{k_{m-n+1(T^{n-1}x)}+1}T^{n-2}x) \\
&\leq q^2.\delta[O_T(T^{n-2}x, m-n+2)] \\
&\leq ........... \\
&\leq q^n.\delta[O_T(x, m-n+n)] \\
d(T^n x, T^m x) &\leq \frac{q^n}{1-q} d(x, Tx)
\end{aligned}
$$

Since

$$\lim_{n\to\infty} q^n = 0$$

$\{T^n x\}$ is a Cauchy sequence in X. Since X is T- Orbitally complete, $\exists\ x^* \in X$ such that

$$\lim_{n\to\infty} T^n x = x^* \tag{3.3}$$

We now show that $x^*$ is a fixed point.

$$
\begin{aligned}
d(x^*, Tx^*) &= d(x^*, T^{n+1}x) + d(T^{n+1}x, Tx^*) \\
d(x^*, Tx^*) &\leq d(x^*, T^{n+1}x) + q.max\{d(T^n x, x^*), d(T^n x, T^{n+1}x), \\
&\quad d(x^*, Tx^*), d(T^n x, Tx^*), d(x^*, T^{n+1}x), G(T^n x, x^*)\}
\end{aligned}
$$

As $n \to \infty$ using (3.3), we get

$$d(x^*, Tx^*) \leq qmax\{d(x^*, Tx^*)\}$$

Which is possible if

$$d(x^*, Tx^*) = 0$$

$$x^* = Tx^*$$

Hence, it assures the existence of a fixed point $x^*$.

**Claim: $x^*$ is unique.** Let if possible $x^*, y^*$ be two fixed points of T.

$$\begin{aligned} d(x^*, y^*) &= d(Tx^*, Ty^*) \\ &\leq qmax\{d(x^*, y^*), d(x^*, Tx^*), d(y^*, Ty^*), d(y^*, Tx^*), d(Ty^*, x^*), G(x^*, y^*)\} \\ &= 0 \end{aligned}$$

$$\Rightarrow x^* = y^*$$

Thus, finally, we conclude uniqueness of $x^*$ .

**4 Conclusion** The existence and uniqueness of a unique fixed point in T-Orbitally complete space has been established with the most light form of contraction map viz. PGA-quasi contraction which a significant contribution. This result also extend the domain of ciric Result.

## References

[1] Abodayeh, K.,Shatanawi,W.,Bataihah, A. and Ansari ,A.H.,Some Fixed Point and Common Fixed Point Results Through -Distance Under Nonlinear Contractions,Gazi University Journal of Science,30(1): 293-302 (2017).

[2] Berinde, V., "General constructive fixed point theorems for Ciri´c-type almost contractions in metric spaces", Carpathian Journal of Mathematics, Vol. 24, Issue. 2, pp. 10 – 19, 2008.

[3] Ciric,L. B., "A generalization of Banach's contraction principle", Proceedings of the American Mathematical Society, Vol. 45, Issue. 2, pp. 267-273 (1974).

[4] Ciric,L. B., "Non-self mappings satisfying non-linear contractive condition with applications", Nonlinear Analysis, Vol. 71, Issue. 7, pp. 2927 – 2935 (2009).

[5] Kreyszig, E., Introductory Functional Analysis with Applications, Wiley Classics Library Edition Published (1989).

[6] Kuman Poom and Dung Nguyen van , A generalization of Ciric Fixed Point theorems, Filomat Vol. 29, Issue. 7, pp. 1549-1556 (2015).

[7] Lakshmikantham V. and Ciri’c, L.,”Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces”, Nonlinear Analysis, Vol. 70, Issue. 12, pp. 4341 – 4349 (2009).