Beta exponentiated SD distribution

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Abstract

In this paper we introduced Beta exponentiated SD Distribution (BESD) with its statistical properties.Some important features for this distribution have been studied which includes,Hazard rate function,reliability function and maximum likelihood estimators.

keywords

SD distribution,Beta exponentiated SD distribution,Moment generating function,Order statistics,Maximum likelihood estimation.

1 Introduction

The SD distribution was studied by Dwivedi et al. [7],with the introduction of two extra parameters we defined Beta Exponentiated SD distribution.This study is done by the taking Eugene et. al. [8] as the base work.The present work is a mathematical

study of beta exponetiated SD distribution(BESD).
Beta exponentiation of distribution is studied by various researchers before,like beta exponentiated mukherji-Islam distribution studied by Siddiqui et al [1],Exponentiated Beta distribution by Nadarajah [2],Beta generalised Exponentiated frechet distribution with it's application is studied by Majdah M.Badar [3] .Cordeiro et al [4] and Hashmi [5]studied beta exponentiated Weibull distribution and Feroze et al [6] studied Beta exponentiated gamma distribution and its properties and many more are there..
We have done beta exponentiation of SD distribution [7] whose *cdf* and *pdf* is given as:

$$F(x) = \theta^{p-1}x^{1-p} \tag{1}$$

$$f(x) = (1-p)x^{-p}\theta^{p-1}; 0 < x < \theta; 0 < p < 1 \tag{2}$$

Where $\theta > 0$ and $p > 0$ are the scale and shape parameters respectively.

Based on the results of Eugene et. al. [8] for the generalised distribution we propose beta exponentiated SD distribution(BESD) with cumulative frequency distribution,probability distribution,reliability function and hazard rate function are give respectively as:

$$F(x) = \frac{1}{B(a,b)}\int_0^{\theta^{p-1}x^{1-p}} w^{a-1}(1-w)^{b-1}dw \tag{3}$$

$$f(x) = \frac{(1-p)}{B(a,b)}(\theta^{p-1}x^{1-p})^{a-1}(1-\theta^{p-1}x^{1-p})^{b-1}x^{-p}\theta^{p-1}, x > 0 \tag{4}$$

$$R(t) = 1 - \frac{1}{B(a,b)}\int_0^{\theta^{p-1}t^{1-p}} w^{a-1}(1-w)^{b-1}dw \tag{5}$$

and

$$h(x) = \frac{f(x)}{1-F(x)} \tag{6}$$

$$= \frac{\frac{1}{B(a,b)}(\theta^{p-1}x^{1-p})^{a-1}1-(\theta^{p-1}x^{1-p})^{b-1}(1-p)\theta^{p-1}x^{-p}}{1-\frac{1}{B(a,b)}\int_0^{\theta^{p-1}x^{1-p}} w^{a-1}(1-w)^{b-1}dw} \tag{7}$$

here,$B(a,b)$ is the beta function with skewness parameters a>0 and kurtosis parameter b>0.

The different shapes of *pdf*,*cdf* and hazard rate function of BESD are studied further. Due to many parameters graph of its statistical properties shows various shapes which can be helpful in various real life data.Some times it is left skewed sometimes right skewed and its shows normal curve too.In reliability analysis in choosing a probability model it is always seen that how hazard rate curve behave when parameter changes because that's why we have done hazard rate shape variation study is also done here which information is useful for the application of the distribution in understanding industrial products.

1.1 Expansion of cumulative distribution function (cdf) of BESD

Cumulative distributive function *cdf* of new Beta exponentiated SD distribution(BESD) is

$$F_{BESD}(x) = \frac{1}{B(a,b)} \int_0^{\theta^{p-1}x^{1-p}} w^{a-1}(1-w)^{b-1} dw \tag{8}$$

In the expansion of *cdf* two cases arises when $b > 0$ is a real non-integer or is an integer.Both cases are studied here,

1.1.1 For $b > 0$ is a real non-integer

$$\begin{aligned} F_{BESD}(x) &= \frac{\Gamma b}{B(a,b)} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\Gamma(b-j)} \int_0^{\theta^{p-1}x^{1-p}} w^{a+j-1} dw \\ &= \frac{\Gamma b}{B(a,b)} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\Gamma(b-j)(a+j)} \left(\theta^{p-1}x^{1-p}\right)^{a+j} \end{aligned} \tag{9}$$

Eq (9) is showing the *cdf* of BESD distribution as an infinite weighted sum of $cdfs$ of SD distributions.

1.1.2 When $b > 0$ is an integer

Similarly, when $b > 0$ is integer than *cdf* od BESD is showing the finite weighted sum of $cdf's$ of SD-distribution,

$$F_{BESD}(x) = \frac{1}{B(a,b)} \sum_{j=0}^{b-1} \binom{b-1}{j} \frac{(-1)^j}{a+j} (\theta^{p-1}x^{1-p})^{a+j} \tag{10}$$

1.1.3 Graphical presentation of *cdf* of BESD

Various graphs of cumulative distribution of BESD shows that with the unit scale parameter, when the value of shape parameter increased the graph changes from left skewed to right skewed and with high scale parameter on a middle value of shape parameter it shows a normal curve.In all cases values of parameters a and b is high.

Figure 1: $\theta = 1, p = 0.1$

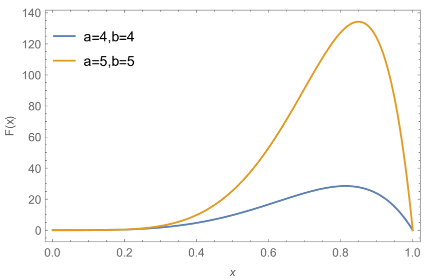


Figure 3: $\theta = 1, p = 0.9$

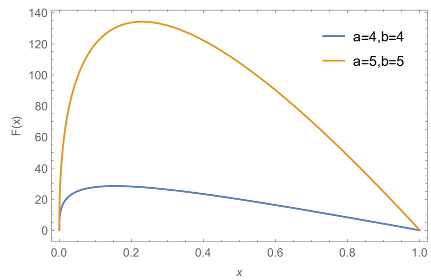


Figure 2: $\theta = 1, p = 0.5$

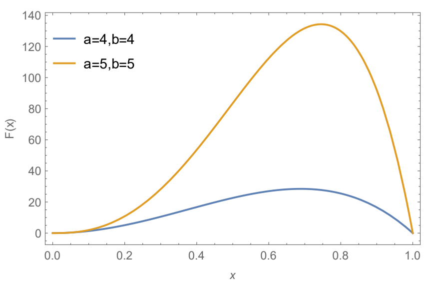
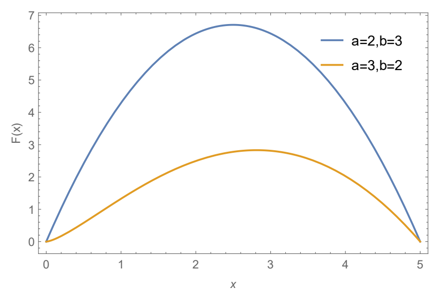


Figure 4: $\theta = 5, p = 0.5$



Graph 1: Cumulative distribution Function

1.2 Expansion of probability density function of BESD

The probability density function of BESD can be expressed as by following equations (11) and (12).

$$f_{BESD}(x) = \frac{1}{B(a,b)} \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma b}{(\Gamma b - j) j!} (G(x))^{a+j-1} g(x)$$

$$f_{BESD}(x) = \frac{(1-p)(\theta^{p-1})^{a+j} \Gamma b}{B(a,b)} \sum_{j=o}^{\infty} \frac{(-1)^j}{(\Gamma b - j) j!} x^{(a+j)(1-p)-1} \tag{11}$$

above is the case when $b > 0$ is a real non-integer and

$$f_{BESD}(x) = \frac{(1-p)(\theta^{p-1})^{a+j} \Gamma b}{B(a,b)} \sum_{j=o}^{\infty} \binom{b-1}{j} (-1)^j x^{(a+j)(1-p)-1} \tag{12}$$

is the case when $b > 0$ is integer.

1.2.1 Graphical presentation of pdf of BESD

Graphs of probability distribution functions of BESD shows various shapes with different values of the parameters.In the first three graphs for the unite scale parameters with different values of the shape parameter and a and b graphs takes various shapes..on the half value of shape parameter p in figure-6 graph shows a normal curve with higher values of a and b..in the figure-8 by keeping p fixed and changing scale parameter θ with higher values of a and b shows the best results of a normal curve.. the next figure-10 shows the curves for the higher value of scale parameter with different values of a and b where shape parameter is fixed..Above shapes shows BESD can be useful on many real life data.

Figure 5: $\theta = 1, p = 0.1$

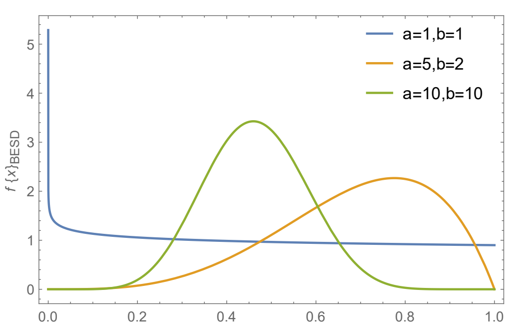


Figure 6: $\theta = 1, p = 0.5$

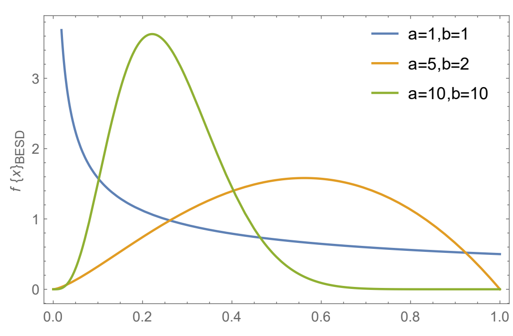


Figure 7: $\theta = 1, p = 0.9$

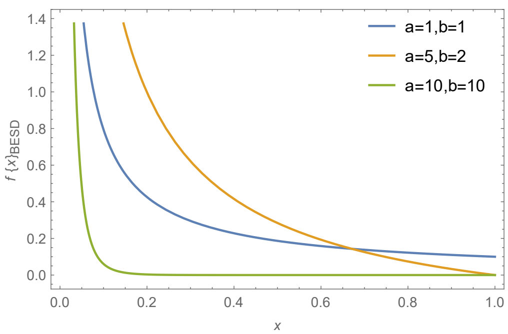


Figure 8: $p = 0.5$

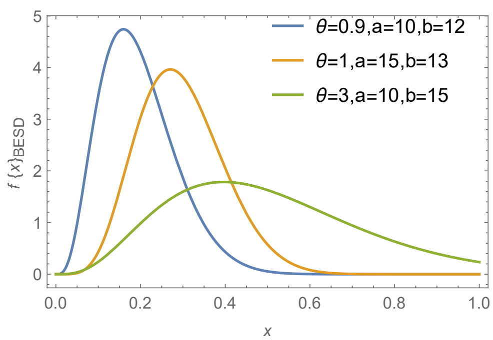


Figure 9: $\theta = 0.9, p = 0.7$

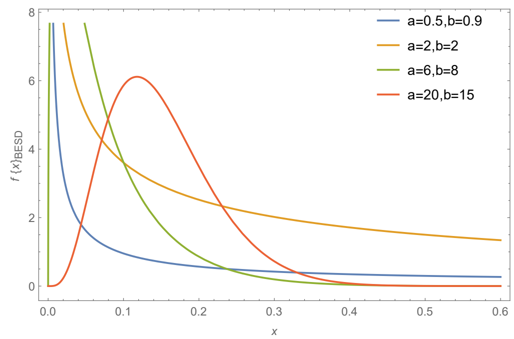
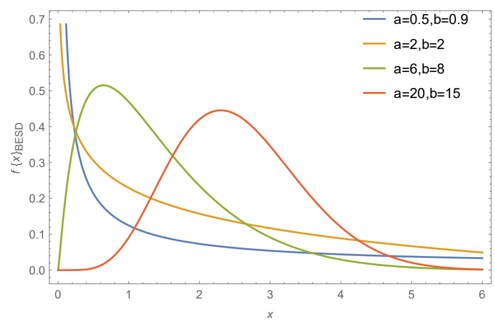


Figure 10: $\theta = 10, p = 0.6$



Graph 2: Probability density Function

1.3 Hazard rate function of BESD

$$h(x) = \frac{f(x)}{1-F(x)} = \frac{\frac{1}{B(a,b)}G(x)^{a-1}(1-G(x))^{b-1}g(x)}{1-\frac{1}{B(a,b)}\int_0^{G(x)} w^{a-1}(1-w)^{b-1}dw} \tag{13}$$

$$= \frac{\frac{1}{B(a,b)}(\theta^{p-1}x^{1-p})^{a-1}(1-\theta^{p-1}x^{1-p})^{b-1}(1-p)\theta^{p-1}x^{-p}}{1-\frac{1}{B(a,b)}\sum_{j=1}^{b-1}\begin{pmatrix} b-1 \\ j \end{pmatrix}\frac{(-1)^j}{(a+j)}(\theta^{p-1}x^{1-p})^{a+j}} \tag{14}$$

for all $\theta > x > 0; 0 < p < 1$

1.3.1 Graphical presentation of Hazard rate function of BESD

Figure 11: $\theta = 1, p = 0.6$

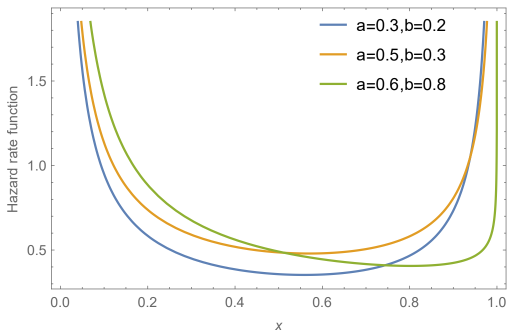
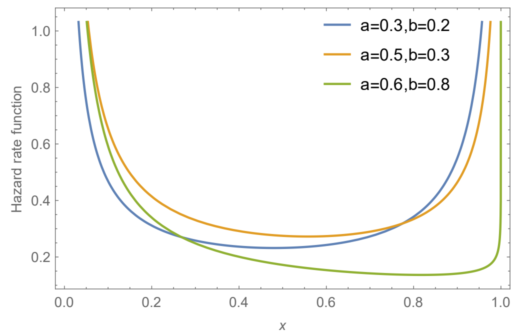


Figure 12: $\theta = 1, p = 0.9$



Graph 3: Hazard Rate Function

Graphs for the hazard rate function shows bathtub shape curves for all the unit scale and $a, b < 1$ values. For all values of $a, b > 1$ graph shows the decreasing failure rate.

1.4 Moment generating function of BESD

The moment generating function mgf of the $BESD$ is given by:

$$M(t) = E(e^{tx})$$
$$M_{BESD}(t) = \int_0^\infty e^{tx} f_{BESD}(x)dx$$
$$= \frac{1-p}{B(a,b)} \int_0^\infty \sum_{j=0}^{b-1} \binom{b-1}{j} (-1)^j e^{tx} (\theta^{p-1})^{a+j} x^{(1-p))(a+j)-1} dx$$

Expanded form of Moment generating function of BESD is,

$$M_{BESD}(t) = \frac{(1-p)\Gamma b}{B(a,b)} \sum_{j=0}^{\infty} \frac{(-1)^{j+(1-p)(a+j)}}{\Gamma(b-j)j!} \frac{\Gamma(1-p)(a+j)\theta^{(p-1)(a+j)}}{t^{(1-p_(a+j)+1)}} \tag{15}$$

for $b > 0$ is real non integer

$$\mu_r^{'} = E(X^r) = \frac{(1-p)\Gamma b}{B(a,b)} \sum_{j=0}^{\infty} \frac{(-1)^{j+(1-p)(a+j)}}{\Gamma(b-j)j!} \theta^{(p-1)(a+j)} \frac{d^r}{dp^r} \frac{\Gamma(1-p)(a+j)}{t^{(1-p)(a+j)}}|_{p=1} \tag{16}$$

eq (16) is for $b > 0$ a real non-integer,

$$\mu_r^{'} = E(X^r) = \frac{(1-p)\Gamma b}{B(a,b)} \sum_{j=0}^{b-1} \binom{b-1}{j=0} \frac{(-1)^{j+(1-p)(a+j)}}{\Gamma(b-j)j!} \theta^{(p-1)(a+j)} \frac{d^r}{dp^r} \frac{\Gamma(1-p)(a+j)}{t^{(1-p)(a+j)}}|_{p=1} \tag{17}$$

eq (17) for $b > 0$ is an integer.

1.5 Moments

With a density function (11) and (12) the rth moment of a random variable x of the BESD distribution is given by

$$\mu_r(x) = E(X^r) \tag{18}$$

$$= \int_0^\infty x^r . f_{BESD(r:n)}\, dx \;\; ; 0 < x < \theta \tag{19}$$

$$= \int_0^\theta x^r . \frac{(1-p)(\theta^{p-1})^{(a+j)}\Gamma b}{B(a,b)} \sum_{j=o}^{\infty} \frac{(-1)^j}{(\Gamma b - j)j!} x^{(a+j)(1-p)-1} dx$$

$as,\ 0 < x < \theta$

$$\mu_r(x) = E(X^r) = \frac{(1-p)\Gamma b}{B(a,b)} \sum_{j=0}^{\infty} \frac{(-1)^j}{\Gamma(b-j)j!} \times \frac{\theta^r}{r + (1-p)(a+j)} \tag{20}$$

results from (20) depicts moments when $b > 0$ is a **real non-integer**.

Similarly,we get moments for $b > 0$ is integer as below by (21),

$$\mu_r(x) = E(X^r) = \frac{(1-p)}{B(a,b)} \sum_{j=0}^{b-1} \binom{b-1}{j} (-1)^j \frac{\theta^r}{r + (1-p)(a+j)} \tag{21}$$

Now,we can find first four moments for both cases,when $b > 0$ is a **real non integer** .

$$\mu_1 = \frac{(1-p)\Gamma b}{B(a,b)} \sum_{j=0}^{\infty} \frac{(-1)^j}{\Gamma(b-j)j!} \times \frac{\theta}{1 + (1-p)(a+j)}$$

$$\mu_2 = \frac{(1-p)\Gamma b}{B(a,b)} \sum_{j=0}^{\infty} \frac{(-1)^j}{\Gamma(b-j)j!} \times \frac{\theta^2}{2 + (1-p)(a+j)}$$

$$\mu_3 = \frac{(1-p)\Gamma b}{B(a,b)} \sum_{j=0}^{\infty} \frac{(-1)^j}{\Gamma(b-j)j!} \times \frac{\theta^3}{3 + (1-p)(a+j)}$$

$$\mu_4 = \frac{(1-p)\Gamma b}{B(a,b)} \sum_{j=0}^{\infty} \frac{(-1)^j}{\Gamma(b-j)j!} \times \frac{\theta^4}{4 + (1-p)(a+j)}$$

Similarly,First four moments for BESD for $b > 0$ is integer,

$$\mu_1(x) = \frac{(1-p)}{B(a,b)} \sum_{j=0}^{b-1} \left(\begin{array}{c} b-1 \\ j \end{array} \right) (-1)^j \frac{\theta}{1+(1-p)(a+j)}$$
$$\mu_2(x) = \frac{(1-p)}{B(a,b)} \sum_{j=0}^{b-1} \left(\begin{array}{c} b-1 \\ j \end{array} \right) (-1)^j \frac{\theta^2}{2+(1-p)(a+j)}$$
$$\mu_3(x) = \frac{(1-p)}{B(a,b)} \sum_{j=0}^{b-1} \left(\begin{array}{c} b-1 \\ j \end{array} \right) (-1)^j \frac{\theta^3}{3+(1-p)(a+j)}$$
$$\mu_4(x) = \frac{(1-p)}{B(a,b)} \sum_{j=0}^{b-1} \left(\begin{array}{c} b-1 \\ j \end{array} \right) (-1)^j \frac{\theta^4}{4+(1-p)(a+j)} \tag{22}$$

The measures of $Skewness$ and $Kurtosis$ of BESD can be obtained as:

$$skew(X) = \frac{\mu_3 - 3\mu_1\mu_2 + 2\mu_1^3}{[\mu_2 - \mu_1^2]^{3/2}} \tag{23}$$
$$kurt(X) = \frac{\mu_4 - 4\mu_1\mu_3 + 6\mu_1^2\mu_2 - 3\mu_1^4}{[\mu_2 - \mu_1^2]^2} \tag{24}$$

1.5.1 Graphical presentation of Skewness and Kurtosis of BESD

Graphical representation of skewness of the BESD distribution shows the various shapes of the graph for the diffrent value of a where other parameter b kept fixed.It shows how skewness varies with the different values of a.In figure-13 value of $theta$ is on lower side whereas in figure-14 it is on higher side.We can easily observe that in both figures BESD showed a right skewed graph.

Similarly graphical presentation of kurtosis of BESD shows the various shapes of the graph for the different value of b where other parameter a is kept fixed.It shows how tails of the graph varies with the different values of b.

Figure 13: $\theta = 3, p = 0.9$

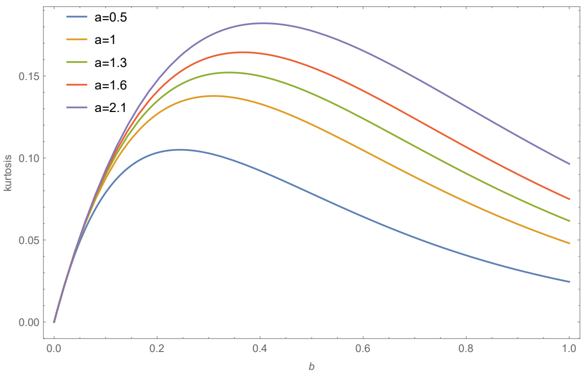
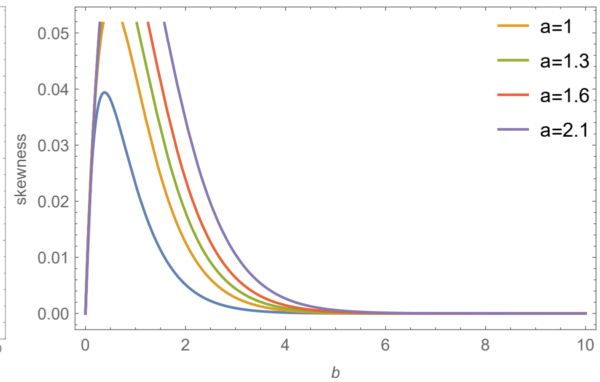


Figure 14: $\theta = 10, p = 0.9$



Graph 4: Skewness

Figure 15: $\theta = 3, p = 0.9$

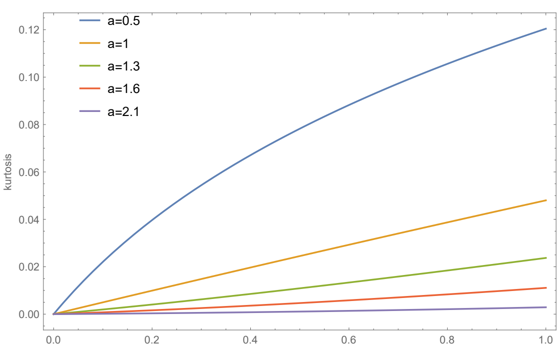
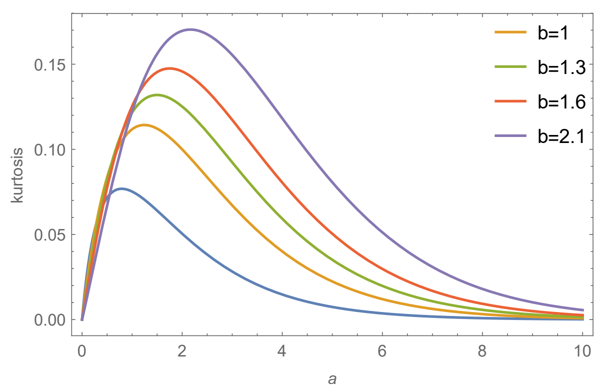


Figure 16: $\theta = 10, p = 0.9$



Graph 5: Kurtosis

2 Order statistics of BESD

L et $X_{1:n}, X_{2:n}, ...X_{n:n}$ denotes the order statistics of BESD distribution corresponding to a random sample $X_1, X_2......X_n$ of th size n of SD distribution.
Based on the results of David and Nadaraja [9] and Arnold et. al [10] the *pdf* of order statistics $X_{r:n}$ of BESD can be defined as

$$f_{BESD(r:n)} = \sum_{i=0}^{n-r}\sum_{j=0}^{b-1}\left(\begin{array}{c} n-r \\ i \end{array}\right)\left(\begin{array}{c} b-1 \\ j \end{array}\right)(-1)^{i+j}\frac{(1-p)\theta^{(p-1)(a+j)(n+i)}}{B(r,n-r+1)[B(a,b)]^{n+j}}\times x^{(a+j)(1-p)(n+j)-1} \tag{25}$$

where $1 \leq r < n$

Equation (26) and (27) shows the *pdf* of the smallest (r=1) and largest (r=n) order statistics of BESD respectively,

$$f_{BESD(1:n)}(x) = n\left[1-\theta^{p-1}x^{1-p}\right]^{n-1}(1-p)x^{-p}\theta^{p-1} \tag{26}$$

$$f_{BESD(n:n)}(x) = n\left[\theta^{p-1}x^{1-p}\right]^{n-1}(1-p)x^{-p}\theta^{p-1} \tag{27}$$

3 Estimation of parameters of BESD

For BESD distribution we take a random sample with $X_1, X_2.....X_n$ of size n.The likelihood function for the vector of parameters $\Phi = (p, \theta, a, b)$ can be written as,

$$\begin{aligned} Lf(x_i, \Phi) &= \prod_{i=1}^{n} f(x_i, \Phi) \\ &= \frac{(1-p)^n\theta^{(p-1)^n}}{[B(a,b)]^n}\Pi_{i=1}^{n}[\theta^{p-1}x_i^{1-p}]^{a-1}\left\{1-\theta^{p-1}x_i^{1-p}\right\}^{b-1} \end{aligned} \tag{28}$$

Taking log-likelihood function for the vectors of parameters $\Phi = (p, \theta, a, b)$,We get

$$logL = nlog(1-p) - nlog[B(a,b)] + (p-1)(n+a-1)log\theta + (1-p)\sum_{i=1}^{n} x_i \tag{29}$$
$$+(b-1)\sum_{i=1}^{n} log(1-\theta^{p-1}x_i^{1-p})$$

$$\frac{\partial logL}{\partial \theta} = \frac{(p-1)(n+a-1)}{\theta} + \frac{(b-1)\sum_{i=1}^{n(1-p)\theta^{p-2}x_i^{1-p}}}{1-\theta^{p-1}x_i^{1-p}} \tag{30}$$

$$\frac{\partial logL}{\partial p} = \frac{(n+a-1)}{\theta} - \sum_{i=1}^{n} x_i - \frac{(b-1)\sum_{i=1}^{n}(\frac{x_i}{\theta})^{1-p} log[1-\frac{x_i}{\theta}]}{[1-(\frac{x_i}{\theta})^{1-p}]} \tag{31}$$

$$\frac{\partial logL}{\partial a} = n\psi(a+b) - n\psi(a) + (p-1)log\theta \tag{32}$$

$$\frac{\partial logL}{\partial b} = n\psi(a+b) - n\psi(b) + \sum_{i=1}^{n} log\left\{1-\theta^{p-1}x_i^{1-p}\right\} \tag{33}$$

where $\psi(.)$ is a diagamma function. Equating above equation to zero we get the estimates of the following unknown parameters by maximum likelihood method:

$$\hat{\theta} = \left[\frac{(n+a-1)}{(n+a+b-2)}\right]^{\frac{1}{p-1}} \tag{34}$$

$$\hat{p} = \frac{2log\theta - log(b-1) - log(n+a-1) - \sum_{i=1}^{n} log[log(1-\frac{x_i}{\theta})]}{log\theta + \sum_{i=1}^{n} logx_i} \tag{35}$$

$$\psi(\hat{a}) = \psi(a+b) + \frac{(p-1)log\theta}{n} \tag{36}$$

$$\psi(\hat{b}) = \psi(a+b) + \frac{\sum_{i=1}^{n} log[1-\theta^{p-1}x_i^{1-p}]}{n} \tag{37}$$

4 Conclusion

Present study, Beta Exponentiated SD distribution, is based on the distribution studies by Dwivedi et al. [7]. The basic idea for this study is taken from work of Eugene et. al. [8].The present work is a mathematical study of BESD, therefore the different mathematical characteristics of the distribution have been evolved. The mathematical development of the present study is very useful for the study of reliability analysis, main functions have been evolved which are useful for the theory of

Reliability Analysis. One of the important statistical studies is the study of the distribution of the ordered observations, here that study has also been done to enlarge the application side of the new distribution.

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