# Effects of variation in salinity due to pollution on the dynamics of two competing species system in aquatic habitat

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## Abstract

In this paper, a mathematical model is proposed to investigate the effects of variation in salinity level due to pollution on the dynamics of two competing species system in an aquatic habitat. In the model formulation, growth rates and the inter-specific competition rates are assumed to be dependent on salt concentration in water. The stability and bifurcation analysis of the model is being carried out and numerical simulation is performed to support the analytical results. From the analysis of the model the criteria for the survival or extinction of the two populations are derived, depending on the conditions involving parameters such as input rate of pollutants, depletion rate of pollutants and input rate of salt due to inflow of pollutants in water. From the analysis it has been also observed that the equilibrium density of one species will be more than the equilibrium density of the other species provided the level of salt concentration is favourable to either of the species.

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## 1. Introduction

Salinity is a measure of the content of salts in water and discharge of waste water from industries, municipalities and runoff of water containing chemicals (fertilizers and pesticides) from agricultural land are introducing salt artificially in water bodies causing increase in its salinity level. Increasing salinity can affect aquatic ecosystems in many ways. It can cause shifts in biotic communities, limit biodiversity, exclude less tolerant species, and cause acute or chronic effects at specific life stages ([2], [10], [23], [26], [24]). Nielsen et al.[15] cited that freshwater aquatic plants are not tolerant to increasing salinity and the upper limit of salinity tolerated by most freshwater aquatic plants appears to be 4000mg/L. They have also observed that at salinities above 1000mg/L, the growth of aquatic plants are reduced. It is established experimentally that high salinity is usually toxic or inhibitory to seed germination of most freshwater plants[22]. There is an example that the germination of seeds from both Sagittaria latifolia and Ruppia megacarpa decreases as salinity level increases[7]. There is another example that if salinity increases, then there is general decrease in abundance and richness of rotifers and microcrustaceans [4]. Barbour[3] studied the effects of competition and salinity on the growth of a salt marsh plant species. To investigate the effect of increasing salinity on two competing species the author considered two species, Jaumea and Lolium and observed that the growth of both species decreased as salinity increased. The author also observed that as salinity increased then the inhibiting effect of competition on Jaumea was declined. Author has shown that the intolerant halophyte is restricted to the salt marsh because it is a poor competitor with vigorous glycophytes that inhabit nearly, non-saline soils. Wang et al.[25] investigated the effects of salinity, sediment type and waterlogging on the performance of two interacting species, Phragmites australis (native) and Spartina alterniflora (alien), and shown the competitive dominance of S. alterniflora occurred under the conditions of high salinity, sandy sediment and full immersion. Further, the authors found that the species P. australis exhibited competitive dominance under the conditions of low salinity and non-immersion. From the analysis carried out in the paper, wang et al. shown that the salinity stress altered the competitive balance between Spartina and Phragmites and found that the Spartina had the competitive dominance at the highest salinity level whereas Phragmites tended to dominate over Spartina at the lowest salinity level although Phragmites performance was reduced at this salinity level. Authors have shown that Spartina was more tolerant of salt stress than Phragmites and both the species coexisted at the salinity level 15%. The growth of Phragmites declined rapidly with increasing salinity, suggesting that Phragmites is relatively more sensitive to salinity than Spartina.

However, several authors ([11], [12], [13], [17], [18], [19], [27]) have studied the effects of different environmental and ecological factors on the dynamics of two competing species system using mathematical models but no one of them has considered explicitly the role of salinity on the dynamical behaviour of two competing populations in their mathematical models.

Thus, in view of the above, in this paper we have studied the effects of salinity varying due to pollutants on the survival or extinction of two competing populations.

## 2. Basic assumptions and mathematical model

Let $P$ denotes the concentration of pollutant in water, $S_w$ denotes concentration of salt in water, $N_1$ and $N_2$ denote the densities of logistically growing competing populations.

Keeping in view of the above considerations, the mathematical model describing the dynamics of the system is governed by the following set of differential equations:

**Mathematical model 1**

$$\begin{aligned}
\frac{dP}{dt} &= Q - d_1P, &(1)\\
\frac{dS_w}{dt} &= S_0 - d_2S_w + \alpha P, &(2)\\
\frac{dN_1}{dt} &= r_1(S_w)N_1 - c_1{N_1}^2 - a_1(S_w)N_1N_2, &(3)\\
\frac{dN_2}{dt} &= r_2(S_w)N_2 - c_2{N_2}^2 - a_2(S_w)N_1N_2. &(4)
\end{aligned}$$

with the initial conditions as:
$N_1(0) > 0, N_2(0) > 0, P(0) \geq 0, S_w(0) > 0$. $a_1(S_w)$ and $a_2(S_w)$ are non-negative continuous functions in both the cases when either $S_w \leq S_n$ or $S_w \geq S_n$
Various parameters of the model are defined in Table 1.

**Case A When $S_w$ is greater than threshold level of salt concentration ($S_n$) then habitat is favourable for $N_1$ species** The function $r_1(S_w)$ is the specific growth rate coefficient of $N_1$ species and it increases when concentration of salt in water is greater than the threshold level of salt concentration ($S_w > S_n$). It is assumed that

$$r_1(S_n) = r_{10}, {r_1}^{'}(S_w) \geq 0 \text{ for } S_w \geq S_n. \tag{5}$$

The function $a_1(S_w)$ is the specific decay rate coefficient of $N_1$ species due to interspecific competition coefficient with $N_2$ species and it decreases when $S_w > S_n$. It is assumed that

$$a_1(S_n) = a_{10}, {a_1}^{'}(S_w) \leq 0 \text{ for } S_w \geq S_n \text{ and } a_1(S_w) \leq a_{10} \text{ for } S_w \geq S_n. \tag{6}$$

The function $r_2(S_w)$ is the specific growth rate coefficient of $N_2$ species and it decreases when concentration of salt in water is greater than the threshold level of salt concentration ($S_w > S_n$). It is assumed that

$$r_2(S_n) = r_{20}, {r_2}^{'}(S_w) \leq 0 \text{ for } S_w \geq S_n. \tag{7}$$

The function $a_2(S_w)$ is the specific decay rate coefficient of $N_2$ species due to interspecific competition coefficient with $N_1$ species and it increases when $S_w > S_n$. It is assumed that

$$a_2(S_n) = a_{20}, {a_2}^{'}(S_w) \geq 0 \text{ for } S_w \geq S_n \text{ and } a_2(S_w) \geq a_{20} \text{ for } S_w \geq S_n. \tag{8}$$

**Case B When $S_w$ is less than threshold level of salt concentration then habitat is favourable for $N_2$ species** The function $r_1(S_w)$ is the specific growth rate coefficient of $N_1$ species and it decreases when concentration of salt in water is less than the threshold level of salt concentration $(S_w < S_n)$. It is assumed that

$$r_1(S_n) = r_{10}, {r_1}^{'}(S_w) \leq 0 \quad \text{for } S_w \leq S_n. \tag{9}$$

The function $a_1(S_w)$ is the specific decay rate coefficient of $N_1$ species due to interspecific competition coefficient with $N_2$ species and it increases when $S_w < S_n$. It is assumed that

$$a_1(S_n) = a_{10}, {a_1}^{'}(S_w) \geq 0 \ \text{ for } S_w \leq S_n \ \text{ and } a_1(S_w) \geq a_{10} \ \text{ for } S_w \leq S_n. \tag{10}$$

The function $r_2(S_w)$ is the specific growth rate coefficient of $N_2$ species and it increases when concentration of salt in water is less than the threshold level of salt concentration $(S_w < S_n)$. It is assumed that

$$r_2(S_n) = r_{20}, {r_2}^{'}(S_w) \geq 0 \quad \text{for } S_w \leq S_n. \tag{11}$$

The function $a_2(S_w)$ is the specific decay rate coefficient of $N_2$ species due to interspecific competition coefficient with $N_1$ species and it decreases when $S_w < S_n$. It is assumed that

$$a_2(S_n) = a_{20}, {a_2}^{'}(S_w) \leq 0 \ \text{ for } S_w \leq S_n \ \text{ and } a_2(S_w) \leq a_{20} \ \text{ for } S_w \leq S_n. \tag{12}$$

## 3. Equilibria of the system and their existence

The equilibrium points of the mathematical model 1 are given by the following set of equations:

$$\begin{aligned} Q - d_1 P &= 0, && (13) \\ S_0 - d_2 S_w + \alpha P &= 0, && (14) \\ r_1(S_w)N_1 - c_1{N_1}^2 - a_1(S_w)N_1N_2 &= 0, && (15) \\ r_2(S_w)N_2 - c_2{N_2}^2 - a_2(S_w)N_1N_2 &= 0. && (16) \end{aligned}$$

Table 1: Description of the parameters for the system (1)-(4)

| ***Parameters*** | ***Description*** |
|---|---|
| $Q$ | Constant input rate of pollutant. |
| $d_1$ | Combined depletion rate of pollutant due to natural and artificial mechanism. |
| $S_0$ | Input of salts in water due to environmental processes. |
| $d_2$ | depletion rate of salts from water. |
| $\alpha$ | Input rate of salts associated with pollutants inflow in water on account of anthropogenic activities. |
| $r_1(S_w)$ | Specific growth rate of $N_1$ species which depend on concentration of salt in water |
| $c_1$ | intraspecific competition coefficient of $N_1$ species |
| $a_1(S_w)$ | coefficient of specific decay rate of species $N_1$ due to interspecific competition coefficient with $N_2$ species dependent on $S_w$ |
| $r_2(S_w)$ | Specific growth rate of $N_2$ species which depend on concentration of salt in water |
| $c_2$ | intraspecific competition coefficient of $N_2$ species |
| $a_2(S_w)$ | coefficient of specific decay rate of species $N_2$ due to interspecific competition coefficient with $N_1$ species dependent on $S_w$ |

We derive the equilibrium points of the mathematical model 1 by solving the equations (13)-(16):

1. **Boundary equilibrium point** $E_{11}(\hat{\hat{P}}, \hat{\hat{S}}_w, \hat{\hat{N}}_1, \hat{\hat{N}}_2)$**:**
   $\hat{\hat{P}} = \frac{Q}{d_1}, \hat{\hat{S}}_w = \frac{S_0 d_1 + \alpha Q}{d_1 d_2}$, $\hat{\hat{N}}_1 = 0$, $N_2 = 0$.
2. **Boundary equilibrium point** $E_{22}(\tilde{\tilde{P}}, \tilde{\tilde{S}}_w, \tilde{\tilde{N}}_1, \tilde{\tilde{N}}_2)$**:**
   $\tilde{\tilde{P}} = \frac{Q}{d_1}, \tilde{\tilde{S}}_w = \frac{S_0 d_1 + \alpha Q}{d_1 d_2}$, $\tilde{\tilde{N}}_1 = 0$, $\tilde{\tilde{N}}_2 = \frac{r_2(\frac{S_0 d_1 + \alpha Q}{d_1 d_2})}{c_2}$.
   The boundary equilibrium point $E_{22}$ exists provided
   $r_2(\frac{S_0 d_1 + \alpha Q}{d_1 d_2}) > 0$.
3. **Boundary equilibrium point** $E_{33}(\bar{\bar{P}}, \bar{\bar{S}}_w, \bar{\bar{N}}_1, \bar{\bar{N}}_2)$**:**
   $\bar{\bar{P}} = \frac{Q}{d_1}, \bar{\bar{S}}_w = \frac{S_0 d_1 + \alpha Q}{d_1 d_2}$, $\bar{\bar{N}}_1 = \frac{r_1(\frac{S_0 d_1 + \alpha Q}{d_1 d_2})}{c_1}$, $\bar{\bar{N}}_2 = 0$.
   The boundary equilibrium point $E_{33}$ exists provided
   $r_1(\frac{S_0 d_1 + \alpha Q}{d_1 d_2}) > 0$.
4. **Interior equilibrium point** $E_44(P^{**}, {S_w}^{**}, N_1^{**}, N_2^{**})$**:**
   $P^{**} = \frac{Q}{d_1}$, ${S_w}^{**} = \frac{S_0 d_1 + \alpha Q}{d_1 d_2}$,
   $N_1^{**} = \frac{c_2 r_1(\frac{S_0 d_1 + \alpha Q}{d_1 d_2}) - a_1(\frac{S_0 d_1 + \alpha Q}{d_1 d_2}) r_2(\frac{S_0 d_1 + \alpha Q}{d_1 d_2})}{c_1 c_2 - a_1(\frac{S_0 d_1 + \alpha Q}{d_1 d_2}) a_2(\frac{S_0 d_1 + \alpha Q}{d_1 d_2})}$,
   $N_2^{**} = \frac{c_1 r_2(\frac{S_0 d_1 + \alpha Q}{d_1 d_2}) - a_2(\frac{S_0 d_1 + \alpha Q}{d_1 d_2}) r_1(\frac{S_0 d_1 + \alpha Q}{d_1 d_2})}{c_1 c_2 - a_1(\frac{S_0 d_1 + \alpha Q}{d_1 d_2}) a_2(\frac{S_0 d_1 + \alpha Q}{d_1 d_2})}$.
   The equilibrium point $E_{44}$ exist if either
   $$\frac{c_2}{a_1(\frac{S_0 d_1 + \alpha Q}{d_1 d_2})} > \frac{r_2(\frac{S_0 d_1 + \alpha Q}{d_1 d_2})}{r_1(\frac{S_0 d_1 + \alpha Q}{d_1 d_2})} > \frac{a_2(\frac{S_0 d_1 + \alpha Q}{d_1 d_2})}{c_1} \tag{17}$$
   or
   $$\frac{c_2}{a_1(\frac{S_0 d_1 + \alpha Q}{d_1 d_2})} < \frac{r_2(\frac{S_0 d_1 + \alpha Q}{d_1 d_2})}{r_1(\frac{S_0 d_1 + \alpha Q}{d_1 d_2})} < \frac{a_2(\frac{S_0 d_1 + \alpha Q}{d_1 d_2})}{c_1} \tag{18}$$
   is satisfied.

**Remark 1.** 1. $\frac{\partial N_1^{**}}{\partial Q} > 0, \frac{\partial N_2^{**}}{\partial Q} < 0$ *for case A.*

2. $\frac{\partial N_1^{**}}{\partial Q} < 0, \frac{\partial N_2^{**}}{\partial Q} > 0$ *for case B.*
   *It is observed that when $S_w > S_n$ then equilibrium level of $N_1$ species increases and the equilibrium level of $N_2$ species decreases with respect to increase in constant input rate of pollutant. If $S_w < S_n$ then equilibrium level of $N_1$ species decreases and the equilibrium level of $N_2$ species increases with respect to increase in input rate of pollutant.*

3. $\frac{\partial N_1^{**}}{\partial \alpha} > 0, \frac{\partial N_2^{**}}{\partial \alpha} < 0$ *for case A.*
4. $\frac{\partial N_1^{**}}{\partial \alpha} < 0, \frac{\partial N_2^{**}}{\partial \alpha} > 0$ *for case B.*

   *It is noticed that when $S_w > S_n$ then equilibrium level of $N_1$ species increases and the equilibrium level of $N_2$ species decreases due to increase in input rate of salts associated with pollutants inflow in water on account of anthropogenic activities. If $S_w < S_n$ then equilibrium level of $N_1$ species decreases and the equilibrium level of $N_2$ species increases due to increase in input rate of salts associated with pollutants inflow in water on account of anthropogenic activities.*
5. $\frac{\partial N_1^{**}}{\partial d_1} < 0, \frac{\partial N_2^{**}}{\partial d_1} > 0$ *for case A.*
6. $\frac{\partial N_1^{**}}{\partial d_1} > 0, \frac{\partial N_2^{**}}{\partial d_1} < 0$ *for case B.*

   *It is observed that when $S_w > S_n$ then equilibrium level of $N_1$ species increases and the equilibrium level of $N_2$ species decreases if the depletion rate of pollutants is increased by artificial mechanisms. If $S_w < S_n$ then equilibrium level of $N_1$ species decreases and the equilibrium level of $N_2$ species increases due to increase in combined depletion rate of pollutants on account of artificial mechanisms.*

## 4. Boundedness of the system

In this section, we obtain sufficient conditions for the boundedness of system (1)-(4). All the solutions of the system given by (1)-(4) for case A will lie in the region $\Omega_1 = \{(P, S_w, N_1, N_2) \in R_+^4 : 0 \le P \le \frac{Q}{d_1}, 0 < S_w \le \frac{S_0 d_1 + \alpha Q}{d_1 d_2}, 0 < N_1 \le \frac{r_1(\frac{S_0 d_1 + \alpha Q}{d_1 d_2})}{c_1}, 0 < N_2 \le \frac{r_{20}}{c_2}\}$ as $t \to \infty$, for all positive initial values $(P(0), S_w(0), N_1(0), N_2(0)) \in R_+^4$.

*Proof.* : From equation (1) of the system, we get

$$\frac{dP}{dt} + d_1 P \le Q.$$

Applying comparison theorem (Hale, 1969 [9]) we obtain

$$0 \le P \le \frac{Q}{d_1} + c_1 e^{-d_1 t}$$

and we have
$\limsup_{t\to\infty} P \leq \frac{Q}{d_1} = P^{**}$.
From equation (2) of the system, we obtain

$\frac{dS_w}{dt} + d_2 S_w \leq (S_0 + \alpha P^{**})$.
Using comparison theorem (Hale, 1969 [9]) we get
$0 \leq S_w \leq \frac{(S_0+\alpha P^{**})}{d_2} + c_2 e^{-d_2 t}$
and we have $\limsup_{t\to\infty} S_w \leq \frac{S_0 d_1 + \alpha Q}{d_1 d_2} = {S_w}^{**}$.

Now, from equation (3) of the mathematical model, we get $\frac{dN_1}{dt} \leq r_1({S_w}^{**})N_1 - c_1{N_1}^2$. On using comparison theorem (Hale, 1969 [9]) we have
$\limsup_{t\to\infty} N_1 \leq \frac{r_1({S_w}^{**})}{c_1} = \frac{r_1(\frac{S_0 d_1+\alpha Q}{d_1 d_2})}{c_1}$.
From equation (4) of the system, we obtain
$\frac{dN_2}{dt} \leq r_{20}N_2 - c_2{N_2}^2$. Using comparison theorem (Hale, 1969 [9]) we have
$\limsup_{t\to\infty} N_2 \leq \frac{r_{20})}{c_2}$.
This completes the proof of the lemma. ☐

All the solutions of the system given by (1)-(4) for case B will lie in the region $\Omega_2 = \{(P, S_w, N_1, N_2) \in R^4_+ : 0 \leq P \leq \frac{Q}{d_1}, 0 < S_w \leq \frac{S_0 d_1+\alpha Q}{d_1 d_2}, 0 < N_1 \leq \frac{r_{10}}{c_1}, 0 < N_2 \leq \frac{r_2(\frac{S_0 d_1+\alpha Q}{d_1 d_2})}{c_2}\}$ as $t \to \infty$, for all positive initial values $(P(0), S_w(0), N_1(0), N_2(0)) \in R^4_+$.

*Proof.* : The proof is similar to that of lemma (4). ☐

## 5. Stability Analysis

Consider the system:

$$\begin{aligned} \frac{dx}{dt} &= f(t,x), && (19)\\ \frac{dy}{dt} &= g(y), && (20) \end{aligned}$$

where, $f$ and $g$ are continuous and locally Lipschitz in $x$ in $R^n$ and the solutions exist for all positive time. Equation 20 is asymptotically autonomous with limit equation 19 if $f(t,x) \to g(x)$ as $t \to \infty$ uniformly for all x in $R^n$.

**Lemma 1.** *([6], [20], [21]) Suppose $e$ be a locally asymptotically stable equilibrium of (20) and $\omega$ be the $\omega$-limit set of a forward bounded solution $x(t)$ of (19). If $\omega$ contains a point $y_0$ such that the solution of (20) with $y(0) = y_0$ converges to $e$ as $t \to \infty$, then $\omega = e$, that is, $x(t) \to e$ as $t \to \infty$.*

**Corollary 1.** *If the solutions of the system (19) are bounded and equilibrium $e$ of the limit system (20) is globally asymptotically stable then any solution $x(t)$ of the system (19) satisfies $x(t) \to e$ as $t \to \infty$.*

Since, we have $P^{**} \leq \limsup_{t\to\infty} P$ and $S_w^*{}^* \leq \limsup_{t\to\infty} S_w$,
then, after solving the differential equation for $P$ and $S_w$, the system (1)-(4) is reduced to the following equivalent autonomous system:

**Mathematical model-2**

$$\frac{dN_1}{dt} = N_1(r_1(S_w^*{}^*)) - c_1{N_1}^2 - a_1(S_w^*{}^*)N_1N_2, \tag{21}$$

$$\frac{dN_2}{dt} = N_2(r_2(S_w^*{}^*)) - c_2{N_2}^2 - a_2(S_w^*{}^*)N_1N_2. \tag{22}$$

### *5.1. **Equilibria of the mathematical model-2 and their existence***

The equilibrium points of the mathematical model-2 (21)-(22) are obtained by solving the following equations:

$$N_1(r_1(S_w^*{}^*)) - c_1{N_1}^2 - a_1(S_w^*{}^*)N_1N_2 = 0, \tag{23}$$

$$N_2(r_2(S_w^*{}^*)) - c_2{N_2}^2 - a_2(S_w^*{}^*)N_1N_2 = 0. \tag{24}$$

We derive the equilibrium points of the mathematical model by solving the equations (23)-(24):

1. **Trivial equilibrium point $E_1(\hat{N}_1, \hat{N}_2)$:**
   $\hat{N}_1 = 0$, $N_2 = 0$.

2. **Boundary equilibrium point** $E_2(\tilde{N_1}, \tilde{N_2})$**:**
   $\tilde{N_1} = 0$, $\tilde{N_2} = \frac{r_2({S_w}^{**})}{c_2} = \frac{r_2(\frac{S_0 d_1 + \alpha Q}{d_1 d_2})}{c_2}$.
   The boundary equilibrium point $E_2$ exists provided
   $r_2(\frac{S_0 d_1 + \alpha Q}{d_1 d_2}) > 0$.
3. **Boundary equilibrium point** $E_3(\bar{N_1}, \bar{N_2})$**:**
   $\bar{N_1} = \frac{r_1({S_w}^{**})}{c_1} = \frac{r_1(\frac{S_0 d_1 + \alpha Q}{d_1 d_2})}{c_1}$, $\bar{N_2} = 0$.
   The boundary equilibrium point $E_3$ exists provided
   $r_1(\frac{S_0 d_1 + \alpha Q}{d_1 d_2}) > 0$.
4. **Interior equilibrium point** $E_4(N_1^*, N_2^*)$**:**
   $N_1^* = \frac{c_2 r_1({S_w}^{**}) - a_1({S_w}^{**}) r_2({S_w}^{**})}{c_1 c_2 - a_1({S_w}^{**}) a_2({S_w}^{**})} = \frac{c_2 r_1(\frac{S_0 d_1 + \alpha Q}{d_1 d_2}) - a_1(\frac{S_0 d_1 + \alpha Q}{d_1 d_2}) r_2(\frac{S_0 d_1 + \alpha Q}{d_1 d_2})}{c_1 c_2 - a_1(\frac{S_0 d_1 + \alpha Q}{d_1 d_2}) a_2(\frac{S_0 d_1 + \alpha Q}{d_1 d_2})}$,
   $N_2^* = \frac{c_1 r_2({S_w}^{**}) - a_2({S_w}^{**}) r_1({S_w}^{**})}{c_1 c_2 - a_1({S_w}^{**}) a_2({S_w}^{**})} = \frac{c_1 r_2(\frac{S_0 d_1 + \alpha Q}{d_1 d_2}) - a_2(\frac{S_0 d_1 + \alpha Q}{d_1 d_2}) r_1(\frac{S_0 d_1 + \alpha Q}{d_1 d_2})}{c_1 c_2 - a_1(\frac{S_0 d_1 + \alpha Q}{d_1 d_2}) a_2(\frac{S_0 d_1 + \alpha Q}{d_1 d_2})}$.
   The equilibrium point $E_4$ exist if either
   $$\frac{c_2}{a_1(\frac{S_0 d_1 + \alpha Q}{d_1 d_2})} > \frac{r_2(\frac{S_0 d_1 + \alpha Q}{d_1 d_2})}{r_1(\frac{S_0 d_1 + \alpha Q}{d_1 d_2})} > \frac{a_2(\frac{S_0 d_1 + \alpha Q}{d_1 d_2})}{c_1} \tag{25}$$
   or
   $$\frac{c_2}{a_1(\frac{S_0 d_1 + \alpha Q}{d_1 d_2})} < \frac{r_2(\frac{S_0 d_1 + \alpha Q}{d_1 d_2})}{r_1(\frac{S_0 d_1 + \alpha Q}{d_1 d_2})} < \frac{a_2(\frac{S_0 d_1 + \alpha Q}{d_1 d_2})}{c_1} \tag{26}$$
   is satisfied.

### *5.2. Dynamical behaviour*

**Theorem 1.** *The equilibrium point* $E_1$ *of the mathematical model 2 is unstable locally if* $r_1(\frac{S_0 d_1 + \alpha Q}{d_1 d_2}) > 0$ *and* $r_2(\frac{S_0 d_1 + \alpha Q}{d_1 d_2}) > 0$ *in both of the cases A and B.*

*Proof.* The Jacobian matrix evaluated at $E_1$ is

$$M_1 = \begin{pmatrix} r_1(S_w^{**}) & 0 \\ 0 & r_2(S_w^{**}) \end{pmatrix}$$

The eigenvalues corresponding to the matrix $M_1$ are $\lambda_1 = r_1(S_w^{**}) = r_1(\frac{S_0 d_1 + \alpha Q}{d_1 d_2})$ and $\lambda_2 = r_2(S_w^{**}) = r_2(\frac{S_0 d_1 + \alpha Q}{d_1 d_2})$. If $\lambda_1 > 0$ and $\lambda_2 > 0$ then equilibrium point $E_1$ is unstable in both of the cases A and B. □

**Theorem 2.** *The equilibrium point $E_1$ of the mathematical model 2 is saddle point in both of the cases A and B provided either $r_1(\frac{S_0d_1+\alpha Q}{d_1d_2}) > 0$ and $r_2(\frac{S_0d_1+\alpha Q}{d_1d_2}) < 0$ or $r_1(\frac{S_0d_1+\alpha Q}{d_1d_2}) < 0$ and $r_2(\frac{S_0d_1+\alpha Q}{d_1d_2}) > 0$ holds good respectively.*

*Proof.* The Jacobian matrix evaluated at $E_1$ is

$$M_{11} = \begin{pmatrix} r_1(S_w^{**}) & 0 \\ 0 & r_2(S_w^{**}) \end{pmatrix}$$

The eigenvalues corresponding to the matrix $M_{11}$ are $\lambda_1 = r_1(S_w^{**}) = r_1(\frac{S_0d_1+\alpha Q}{d_1d_2})$ and $\lambda_2 = r_2(S_w^{**}) = r_2(\frac{S_0d_1+\alpha Q}{d_1d_2})$. If $\lambda_1 > 0$ and $\lambda_2 < 0$ then equilibrium point $E_1$ is saddle point in the case A. If $\lambda_1 < 0$ and $\lambda_2 > 0$ then equilibrium point $E_1$ is saddle point in the case B. □

**Theorem 3.** *For the mathematical model 2 given by (21)-(22) we find that $E_2$ is asymptotically stable in the case A if $r_2(\frac{S_0d_1+\alpha Q}{d_1d_2}) > 0$ and $\frac{c_2}{r_2(\frac{S_0d_1+\alpha Q}{d_1d_2})} < \frac{a_1(\frac{S_0d_1+\alpha Q}{d_1d_2})}{r_1(\frac{S_0d_1+\alpha Q}{d_1d_2})}$ are satisfied.*

*Proof.* The Jacobian matrix evaluated at $E_2$ is

$$M_2 = \begin{pmatrix} r_1(S_w^{**}) - a_1(S_w^{**})\tilde{N}_2 & 0 \\ -a_2(S_w^{**})\tilde{N}_2 & -c_2\tilde{N}_2 \end{pmatrix}$$

The eigenvalues corresponding to the matrix $M_2$ are
$\lambda_1 = -r_2(S_w^{**}) = -r_2(\frac{S_0d_1+\alpha Q}{d_1d_2})$ and
$\lambda_2 = r_1(S_w^{**}) - \frac{a_1(S_w^{**})r_2(S_w^{**})}{c_2} = r_1(\frac{S_0d_1+\alpha Q}{d_1d_2}) - \frac{a_1(\frac{S_0d_1+\alpha Q}{d_1d_2})r_2(\frac{S_0d_1+\alpha Q}{d_1d_2})}{c_2}$.
In the case A $\lambda_1 < 0$ and $\lambda_2 < 0$, if $r_2(\frac{S_0d_1+\alpha Q}{d_1d_2}) > 0$ and
$\frac{c_2}{r_2(\frac{S_0d_1+\alpha Q}{d_1d_2})} < \frac{a_1(\frac{S_0d_1+\alpha Q}{d_1d_2})}{r_1(\frac{S_0d_1+\alpha Q}{d_1d_2})}$ then $E_2$ is locally asymptotically stable. □

**Remark 2.** *$N_2$ species will survive and $N_1$ species will tend to extinction if the ratio of interspecific competition rate of $N_1$ species to its growth rate is greater than the ratio of intraspecific competition rate of $N_2$ species to its growth rate.*

**Theorem 4.** *For the mathematical model 2 given by (21)-(22) we find that $E_2$ is asymptotically stable in the case B if either $r_1(\frac{S_0d_1+\alpha Q}{d_1d_2}) < 0$ or if $\frac{c_2}{r_2(\frac{S_0d_1+\alpha Q}{d_1d_2})} < \frac{a_1(\frac{S_0d_1+\alpha Q}{d_1d_2})}{r_1(\frac{S_0d_1+\alpha Q}{d_1d_2})}$ are satisfied when $r_1(\frac{S_0d_1+\alpha Q}{d_1d_2}) > 0$.*

*Proof.* The Jacobian matrix evaluated at $E_2$ is

$$M_{22} = \begin{pmatrix} r_1(S_w^{*}{}^{*}) - a_1(S_w^{*}{}^{*})\tilde{N}_2 & 0 \\ -a_2(S_w^{*}{}^{*})\tilde{N}_2 & -c_2\tilde{N}_2 \end{pmatrix}$$

The eigenvalues corresponding to the matrix $M_{22}$ are
$\lambda_1 = -r_2(S_w^{*}{}^{*}) = -r_2(\frac{S_0d_1+\alpha Q}{d_1d_2})$ and
$\lambda_2 = r_1(S_w^{*}{}^{*}) - \frac{a_1(S_w^{*}{}^{*})r_2(S_w^{*}{}^{*})}{c_2} = r_1(\frac{S_0d_1+\alpha Q}{d_1d_2}) - \frac{a_1(\frac{S_0d_1+\alpha Q}{d_1d_2})r_2(\frac{S_0d_1+\alpha Q}{d_1d_2})}{c_2}$.
In the case B $\lambda_1 < 0$ and $\lambda_2 < 0$, if either $r_1(\frac{S_0d_1+\alpha Q}{d_1d_2}) < 0$ or if $\frac{c_2}{r_2(\frac{S_0d_1+\alpha Q}{d_1d_2})} < \frac{a_1(\frac{S_0d_1+\alpha Q}{d_1d_2})}{r_1(\frac{S_0d_1+\alpha Q}{d_1d_2})}$ in the case B when $r_1(\frac{S_0d_1+\alpha Q}{d_1d_2}) > 0$, then $E_2$ is locally asymptotically stable. ☐

**Theorem 5.** *For the mathematical model 2 given by (21)-(22) we find that* $E_3$ *is asymptotically stable in the case A if either* $r_2(\frac{S_0d_1+\alpha Q}{d_1d_2}) < 0$ *or if* $\frac{c_1}{r_1(\frac{S_0d_1+\alpha Q}{d_1d_2})} < \frac{a_2(\frac{S_0d_1+\alpha Q}{d_1d_2})}{r_2(\frac{S_0d_1+\alpha Q}{d_1d_2})}$ *when* $r_2(\frac{S_0d_1+\alpha Q}{d_1d_2}) > 0$.

*Proof.* The Jacobian matrix evaluated at $E_3$ is

$$M_3 = \begin{pmatrix} -c_1N_1 & -a_1(S_w^{*}{}^{*})\bar{N}_1 \\ 0 & r_2(S_w^{*}{}^{*}) - a_2(S_w^{*}{}^{*})\bar{N}_1 \end{pmatrix}$$

The eigenvalues corresponding to the matrix $M_3$ are
$\lambda_1 = -r_1(S_w^{*}{}^{*}) = -r_1(\frac{S_0d_1+\alpha Q}{d_1d_2})$ and
$\lambda_2 = r_2(S_w^{*}{}^{*}) - \frac{a_2(S_w^{*}{}^{*})r_1(S_w^{*}{}^{*})}{c_1} = r_2(\frac{S_0d_1+\alpha Q}{d_1d_2}) - \frac{a_2(\frac{S_0d_1+\alpha Q}{d_1d_2})r_1(\frac{S_0d_1+\alpha Q}{d_1d_2})}{c_1}$.
For case A $\lambda_1 < 0$ and $\lambda_2 < 0$ if either $r_2(\frac{S_0d_1+\alpha Q}{d_1d_2}) < 0$ or if $\frac{c_1}{r_1(\frac{S_0d_1+\alpha Q}{d_1d_2})} < \frac{a_2(\frac{S_0d_1+\alpha Q}{d_1d_2})}{r_2(\frac{S_0d_1+\alpha Q}{d_1d_2})}$ in the case A when $r_2(\frac{S_0d_1+\alpha Q}{d_1d_2}) > 0$ then $E_3$ is asymptotically stable. ☐

**Remark 3.** $N_1$ *species will survive and* $N_2$ *species will tend to extinction if the ratio of interspecific competition rate of* $N_2$ *species to its growth rate is greater than the ratio of intraspecific competition rate of* $N_1$ *species to its growth rate.*

**Theorem 6.** *For the mathematical model 2 given by (21)-(22) we find that* $E_3$ *is asymptotically stable in the case B if* $r_1(\frac{S_0d_1+\alpha Q}{d_1d_2}) > 0$ *and* $\frac{c_1}{r_1(\frac{S_0d_1+\alpha Q}{d_1d_2})} < \frac{a_2(\frac{S_0d_1+\alpha Q}{d_1d_2})}{r_2(\frac{S_0d_1+\alpha Q}{d_1d_2})}$ *are satisfied.*

*Proof.* The Jacobian matrix evaluated at $E_3$ is

$$M_{33} = \begin{pmatrix} -c_1N_1 & -a_1(S_w^{*}{}^{*})\bar{N}_1 \\ 0 & r_2(S_w^{*}{}^{*}) - a_2(S_w^{*}{}^{*})\bar{N}_1 \end{pmatrix}$$

The eigenvalues corresponding to the matrix $M_{33}$ are
$\lambda_1 = -r_1(S_w^{*}{}^{*}) = -r_1(\frac{S_0d_1+\alpha Q}{d_1d_2})$ and
$\lambda_2 = r_2(S_w^{*}{}^{*}) - \frac{a_2(S_w^{*}{}^{*})r_1(S_w^{*}{}^{*})}{c_1} = r_2(\frac{S_0d_1+\alpha Q}{d_1d_2}) - \frac{a_2(\frac{S_0d_1+\alpha Q}{d_1d_2})r_1(\frac{S_0d_1+\alpha Q}{d_1d_2})}{c_1}$.
For case B $\lambda_1 < 0$ and $\lambda_2 < 0$ if $r_1(\frac{S_0d_1+\alpha Q}{d_1d_2}) > 0$ and
$\frac{c_1}{r_1(\frac{S_0d_1+\alpha Q}{d_1d_2})} < \frac{a_2(\frac{S_0d_1+\alpha Q}{d_1d_2})}{r_2(\frac{S_0d_1+\alpha Q}{d_1d_2})}$ then $E_3$ is asymptotically stable. □

**Theorem 7.** *The equilibrium point $E_4$ of mathematical model 2 is always stable under the existence condition (25) for interior equilibrium point and unstable under the condition (26).*

*Proof.* The Jacobian matrix evaluated at $E_4$ is

$$M_4 = \begin{pmatrix} -c_1N_1^* & -a_1(S_w^{*}{}^{*})N_1^* \\ -a_2(S_w^{*}{}^{*})N_2^* & -c_2N_2^* \end{pmatrix}$$

The eigenvalues corresponding to the matrix $M_{33}$ are given by the following characteristic equation

$$\lambda^2 + \lambda(c_1N_1^* + c_2N_2^*) + (c_1c_2 - a_1(S_w^{*}{}^{*})a_2(S_w^{*}{}^{*})) = 0. \tag{27}$$

Using Hurwitz criteria in (27), we find that the two eigenvalues of the matrix $M_4$ are negative or have negative real part under the existence condition (25) for interior equilibrium point.
Hence, proving the lemma.

### *5.3.* ***Global Stability***

**Theorem 8.** *Equilibrium point $E_4$ is globally asymptotically stable in the positive quadrant of $N_1 - N_2$ plane.*

*Proof.* Let
$W(N_1, N_2) = \frac{1}{N_1N_2}$,

$f_1(N_1, N_2) = N_1(r_1(S_w^{**})) - c_1{N_1}^2 - a_1(S_w^{**})N_1N_2,$
$f_2(N_1, N_2) = N_2(r_2(S_w^{**})) - c_2{N_2}^2 - a_2(S_w^{**})N_1N_2.$
Here, $W(N_1, N_2) > 0$ if $N_1 > 0$ and $N_2 > 0$ in the positive quadrant of $N_1 - N_2$ plane.
Then, we have
$\Delta(N_1, N_2) = \frac{\partial}{\partial N_1}(f_1W) + \frac{\partial}{\partial N_2}(f_2W)$
$\Delta(N_1, N_2) = -\left(\frac{c_1N_1+c_2N_2}{N_1N_2}\right).$
Since, $\Delta(N_1, N_2) < 0$ for $N_1 > 0$ and $N_2 > 0$ and also it is not identically zero in the positive quadrant of $N_1 - N_2$ plane. Therefore, by Bendixson-Dulac criterion, there will be no periodic orbit in the positive quadrant of $N_1 - N_2$ plane. Hence, $E_4$ is globally asymptotically stable in the positive quadrant of $N_1 - N_2$ plane. ☐

### 5.4. ***Local Bifurcation Study***

**Theorem 9** ([1], [8], [14], [16]). *As the parameter* $c_1$ *passes through the value of* $c_1^* = \frac{a_1(S_w^{**})a_2(S_w^{**})}{c_2}$ *then the limiting system given by mathematical model 2 has a saddle node bifurcation around the interior equilibrium point* $E_4$*. However, the transcritical and pitchfork bifurcation can not occur in this system.*

*Proof.* The Jacobian matrix evaluated at $E_4$ is given by

$$J_4{}^* = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Where, $a_{11} = -c_1N_1^*, a_{12} = -a_1(S_w^{**})N_1^*, a_{21} = -a_2(S_w^{**})N_2^*, a_{22} = -c_2N_2^*.$
It is noticed that as the parameter $c_1$ passes through $c_1^*$ then the interior equilibrium point $E_4$ becomes a nonhyperbolic point and the jacobian matrix of mathematical model A at this point will be $J_4{}^* = (a_{ij})_{2*2}$, where $a_{ij}$ as defined above with $a_{11} = -\frac{a_1(S_w^{**})a_2(S_w^{**})}{c_2}{N_1}^*$. $J_4{}^*$ has simple zero eigenvalue $\lambda_1 = 0$, with the other eigenvalue $\lambda_2 = -(c_2{N_2}^* + \frac{a_1(S_w^{**})a_2(S_w^{**})N_1^*}{c_2}$.
Let

$$V_1 = \begin{pmatrix} u_{21} \\ u_{22} \end{pmatrix}$$

and

$$W_1 = \begin{pmatrix} w_{21} \\ w_{22} \end{pmatrix}$$

be the eigenvectors corresponding to the eigenvalue $\lambda_1 = 0$ for ${J_4}^*$ and ${J_4}^{*T}$, respectively, then one can calculate

$$V_1 = \begin{pmatrix} a_3 u_{22} \\ u_{22} \end{pmatrix}$$

where $u_{22} \neq 0$ and $a_3 = -\frac{a_{12}}{a_{11}}$.

$$W_1 = \begin{pmatrix} a_4 w_{22} \\ w_{22} \end{pmatrix}$$

where $w_{22} \neq 0$ and $a_4 = -\frac{a_{21}}{a_{11}}$. Moreover, by rewrite system (21)-(22) in the form $X' = F(X)$, where

$$X = \begin{pmatrix} N_1 \\ N_2 \end{pmatrix}$$

and

$$F(X) = \begin{pmatrix} f_1(N_1, N_2) \\ f_2(N_1, N_2) \end{pmatrix}$$

then it is observed that:

$$\frac{\partial F}{\partial c_1} = F_{c_1} = \begin{pmatrix} -{N_1}^2 \\ 0 \end{pmatrix},$$

$\mathrm{F}_{c_1}(E_4, c_1^*) = \begin{pmatrix} -{N_1}^{*2} \\ 0 \end{pmatrix}$ ,which gives that ${W_1}^T F_{c_1}(E_4, c_1^*) = -a_4 w_{22} {N_1}^{*2} \neq 0$. Thus, transcritical and pitchfork bifurcation cannot occur while the first condition of saddle node bifurcation is satisfied. Since the second derivative of $F$ with respect to $X$ at $(E_4, c_1^*)$ is

$$D^2F(E_4, c_1^*)(U_1, U_1) = \begin{pmatrix} -2c_1{a_3}^2{u_{22}}^2 - 2a_1({S_w}^{**})a_3{u_{22}}^2 \\ -2c_2{u_{22}}^2 - 2a_2({S_w}^{**}){u_{22}}^2 \end{pmatrix}.$$

Therefore, ${W_1}^T[D^2F(E_4, c_1^*)(U_1, U_1)] = (-2c_1{a_3}^2 - 2a_1({S_w}^{**})a_3){u_{22}}^2 a_4 w_{22} + (-2c_2 - 2a_2({S_w}^{**})){u_{22}}^2 w_{22} \neq 0$.

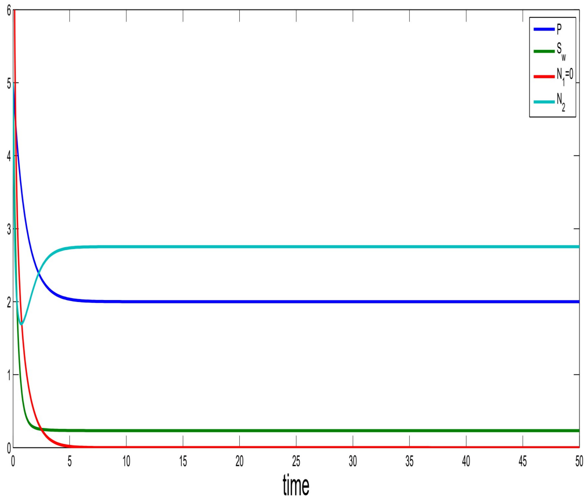

Figure 1: Trajectories of the mathematical model showing the stability behavior of equilibrium point $E_2$.

Hence, by Sotomayor's theorem the occurrence of saddle node bifurcation is proved. $\square$

**Remark 4.** *The limiting system given by mathematical model 2 does not show any type of local bifurcation at the equilibrium points* $E_1$*,* $E_2$ *and* $E_3$*.*

## 6. Numerical simulation

In this section, numerical simulation is done to support the analytical results for the system (1)-(4) by considering the following functional forms:

$$\begin{aligned} r_1(S_w) &= r_{10} + r_{11}(S_w - S_n), \\ r_2(S_w) &= r_{20} - r_{21}(S_w - S_n), \\ a_1(S_w) &= a_{10} - a_{11}(S_w - S_n), \\ a_2(S_w) &= a_{20} + a_{21}(S_w - S_n), \end{aligned} \tag{28}$$

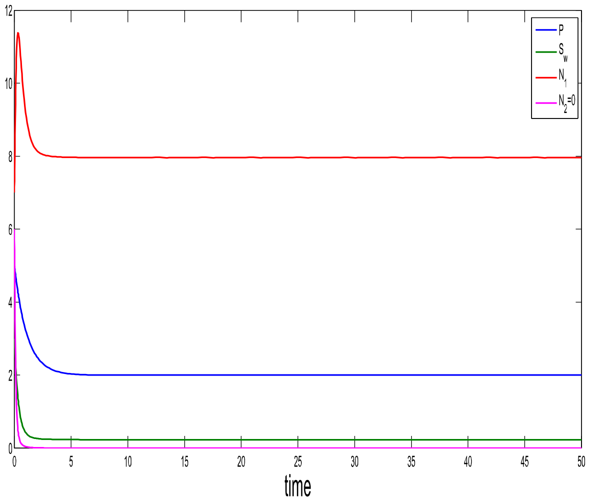

Figure 2: Trajectories of the mathematical model showing the stability behavior of equilibrium point $E_3$.

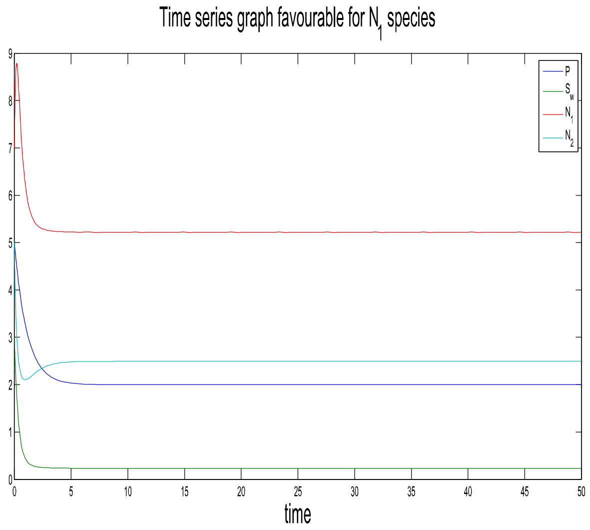

Figure 3: Trajectories of the mathematical model showing the stability behavior of equilibrium point $E_4$ in the case A when $S_w > S_n$.

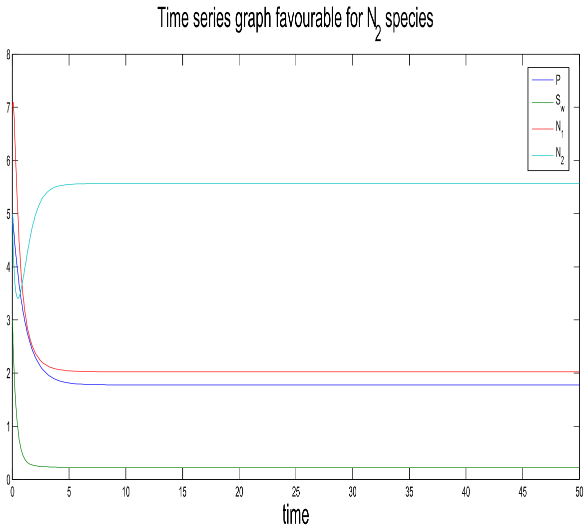

Figure 4: Trajectories of the mathematical model showing the stability behavior of equilibrium point $E_4$ in the case B when $S_w < S_n$.

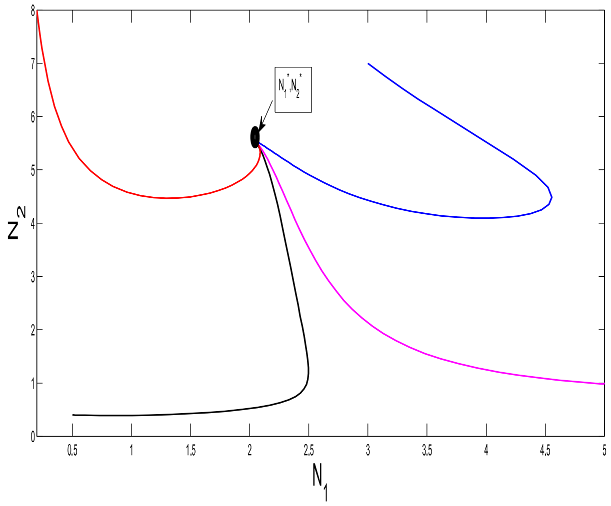

Figure 5: Global stability in $N_1 - N_2$ plane.

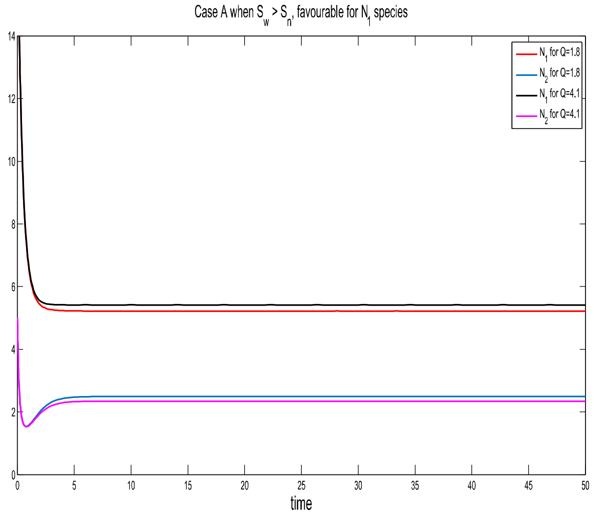

Figure 6: Equilibrium densities of $N_1$ and $N_2$ species for different values of $Q$ in the case A when $S_w > S_n$.

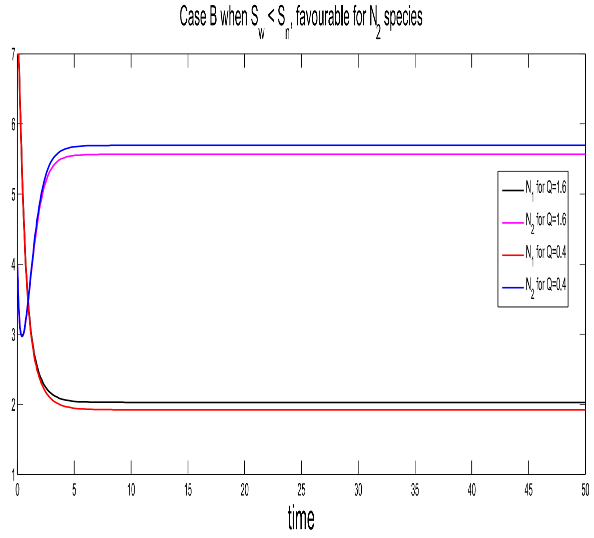

Figure 7: Equilibrium densities of $N_1$ and $N_2$ species for different values of $Q$ in the case B when $S_w < S_n$.

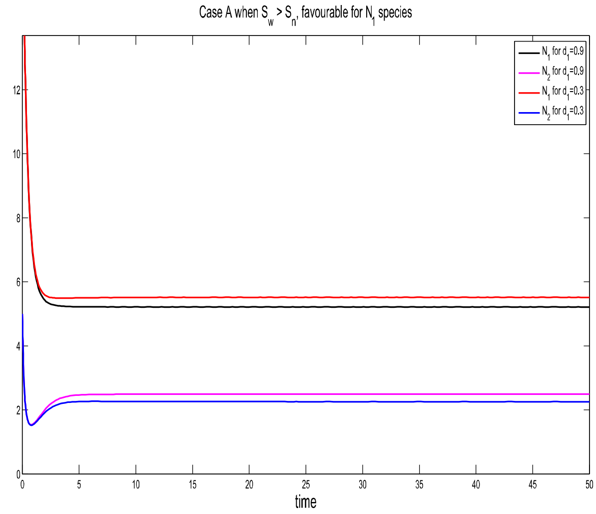

Figure 8: Equilibrium densities of $N_1$ and $N_2$ species for different values of $d_1$ in the case A when $S_w > S_n$.

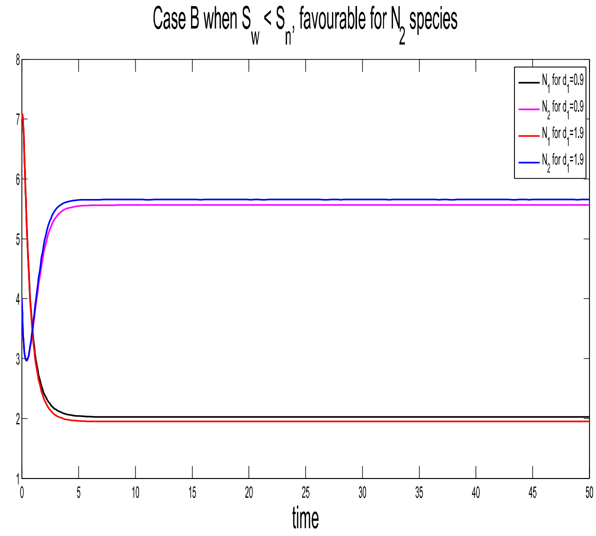

Figure 9: Equilibrium densities of $N_1$ and $N_2$ species for different values of $d_1$ in the case B when $S_w < S_n$.

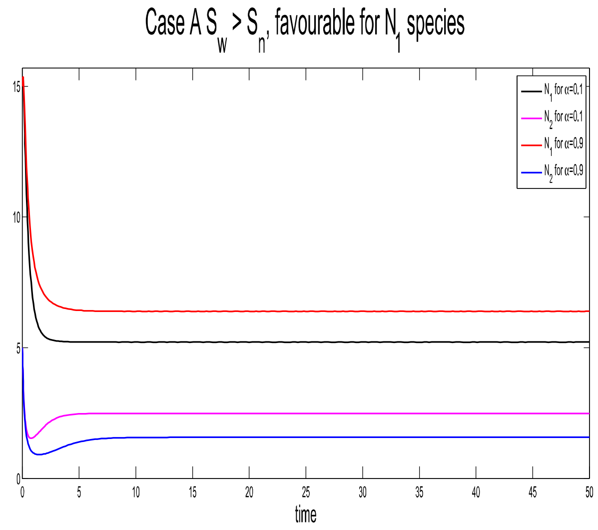

Figure 10: Equilibrium densities of $N_1$ and $N_2$ species for different values of $\alpha$ in the case A when $S_w > S_n$.

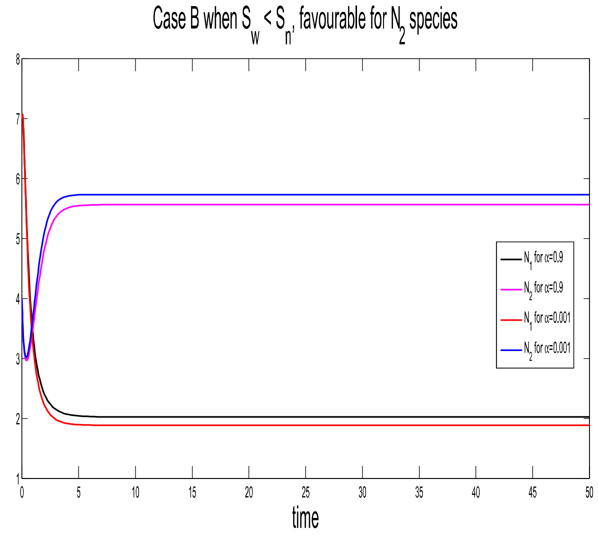

Figure 11: Equilibrium density of $N_1$ and $N_2$ species for different values of $\alpha$ in the case B when $S_w < S_n$.

The set of parametric values for the interior equilibrium point $E_{44}\left(P^{**}, S_w^{**}, {N_1}^{**}, {N_2}^{**}\right)$ in the case A when $S_w > S_n$ is

$$\begin{aligned} &Q = 1.8, d_1 = 0.9, S_0 = 0.50, d_2 = 3.0, \alpha = 0.1, r_{10} = 3.0,\\ &r_{11} = 1.20, S_n = 0.08, c_1 = 0.6, a_{10} = 0.03, a_{11} = 0.056,\\ &r_{20} = 1.5, r_{21} = 0.8035, c_2 = 0.5, a_{20} = 0.025, a_{21} = 0.0002. \end{aligned} \tag{29}$$

For the above mentioned set of functional forms and parametric values, it is noticed that the conditions for the existence of interior equilibrium point $E_{44}\left(P^{**}, S_w^{**}, {N_1}^{**}, {N_2}^{**}\right)$ are satisfied and the equilibrium values are $E_{44}(2.0, 0.233, 5.217, 2.492)$. From the numerical simulation it is observed that the conditions for the boundedness and stability of the interior equilibrium point $E_{44}$ are satisfied which shows that $E_{44}$ is asymptotically stable in case A for the set of parametric values given in (29) (see fig. 3).
Now, we assume following set of parametric values for the interior equilibrium point $E_{44}\left(P^{**}, S_w^{**}, {N_1}^{**}, {N_2}^{**}\right)$ in the case B when $S_w < S_n$

$$\begin{aligned} &Q = 1.6, d_1 = 0.9, S_0 = 0.50, d_2 = 3.0, \alpha = 0.1, r_{10} = 2.0,\\ &r_{11} = 1.20, S_n = 0.5, c_1 = 0.7, a_{10} = 0.03, a_{11} = 0.056,\\ &r_{20} = 1.5, r_{21} = 0.8035, c_2 = 0.3, a_{20} = 0.025, a_{21} = 0.0002. \end{aligned} \tag{30}$$

By using (28) and the set of parametric values defined in (30), it is observed that the conditions for the existence, boundedness and stability of interior equilibrium point $E_{44}\left(P^{**}, S_w^{**}, {N_1}^{**}, {N_2}^{**}\right)$ are satisfied and the equilibrium values are $E_{44}\left(1.7778, 0.2259, 2.0267, 5.5656\right)$ which shows that $E_{44}$ is asymptotically stable in case B for the set of parametric values given in (30) (see fig. 4). Further, numerical simulation is also performed to show the global stability of interior equilibrium point $E_{44}$ (see fig.5).

The boundary equilibrium point $E_{22}(2, 0.2334, 0, 2.7536)$ of the mathematical model is locally asymptotically stable (see fig. 1) for the following set of

parameters:
$Q = 1.8, d_1 = 0.9, S_0 = 0.50, d_2 = 3.0, \alpha = 0.1, r_{10} = 1.0, r_{11} = 0.8, S_n = 0.08, c_1 = 0.9, a_{10} = 0.8, a_{11} = 0.01, r_{20} = 1.5, r_{21} = 0.8035, c_2 = 0.5, a_{20} = 0.025, a_{21} = 0.0002$

The boundary equilibrium point $E_{33}(2.000, 0.2333, 7.9596, 0.0000)$ of the mathematical model is locally asymptotically stable (see fig. 2) for the following set of parametric values:

$Q = 1.8, d_1 = 0.9, S_0 = 0.50, d_2 = 3.0, \alpha = 0.1, r_{10} = 3.0, r_{11} = 1.2, S_n = 0.08, c_1 = 0.4, a_{10} = 0.03, a_{11} = 0.056, r_{20} = 1.5, r_{21} = 0.8035, c_2 = 0.52, a_{20} = 0.3, a_{21} = 0.2.$

## 7. Sensitivity Analysis

Sensitivity index of a variable with respect to parameter is calculated by normalized forward sensitivity method used by Chitnis et al.[5]. Sensitivity index of the interior equilibrium points $E_{44}$ in both the cases A and B are shown in Table 2 and 3 for the parametric values mentioned in (29) and (30) respectively. The sensitivity index table indicates that the equilibrium density of one species is more than the equilibrium density of other species depending significantly on the parameters $Q, \alpha$ and $d_1$.

## 8. Conclusion

In this paper, we have proposed and analyzed a nonlinear mathematical model to investigate mathematically the effects of varying salinity level on the dynamics of two competing populations. From the stability analysis of boundary equilibrium point $E_2$, it is noted that $N_2$ species will survive and $N_1$ species will tend to extinction if the ratio of interspecific competition rate of $N_1$ species to its growth rate is greater than the ratio of intraspecific competition rate of $N_2$ species to its growth rate. It is also observed from the stability analysis of

| Parameters($x_j$) | $\gamma_{x_j}^{P^{**}}$ | $\gamma_{x_j}^{S_w{}^{**}}$ | $\gamma_{x_j}^{N_1{}^{**}}$ | $\gamma_{x_j}^{N_2{}^{**}}$ |
|---|---|---|---|---|
| $Q$ | 1 | 0.285714 | 0.0293123 | -0.046112 |
| $d_1$ | -1 | -0.285714 | -0.0293123 | 0.046112 |
| $S_0$ | 0 | 0.714286 | 0.0732807 | -0.11528 |
| $d_2$ | 0 | -1 | -0.102593 | 0.161392 |
| $\alpha$ | 0 | 0.285714 | 0.0293123 | -0.046112 |
| $r_{10}$ | 0 | 0 | 0.959989 | -0.100608 |
| $r_{11}$ | 0 | 0 | 0.0588793 | -0.00617064 |
| $S_n$ | 0 | 0 | -0.0351747 | 0.0553344 |
| $c_1$ | 0 | 0 | -1.00179 | 0.104989 |
| $a_{10}$ | 0 | 0 | -0.0239266 | 0.00250755 |
| $a_{11}$ | 0 | 0 | 0.00684834 | -0.000717716 |
| $r_{20}$ | 0 | 0 | -0.0205566 | 1.20582 |
| $r_{21}$ | 0 | 0 | 0.00168842 | -0.0990407 |
| $c_2$ | 0 | 0 | 0.0170783 | -1.00179 |
| $a_{20}$ | 0 | 0 | 0.00178764 | -0.10486 |
| $a_{21}$ | 0 | 0 | $2.19284*10^{-6}$ | -0.000128629 |

Table 2: Sensitivity indices of the state variables at the equilibrium point $E_{44}$ to the parameters $(x_j)$ of the model in the case A where $\gamma$=sensitivity index

boundary equilibrium point $E_3$ that $N_1$ species will survive and $N_2$ species will tend to extinction if the ratio of interspecific competition rate of $N_2$ species to its growth rate is greater than the ratio of intraspecific competition rate of $N_1$ species to its growth rate. Coexistence of both the species is ensured from the stability of interior equilibrium point if the condition given by (25) is satisfied. The numerical simulation using hypothetical data is carried out to support the analytical results and graphs are drawn to show the dynamical behaviour of species in the underlying system. The figs. (1-4) shows the stability behaviour of equilibrium points. Now, from fig (6) it is seen that as the input rate of pollutant (Q) increases in the case A ($S_w > S_n$) then the equilibrium density of $N_1$ species

| Parameters($x_j$) | $\gamma_{x_j}^{P^{**}}$ | $\gamma_{x_j}^{{S_w}^{**}}$ | $\gamma_{x_j}^{{N_1}^{**}}$ | $\gamma_{x_j}^{{N_2}^{**}}$ |
|---|---|---|---|---|
| $Q$ | 1 | 0.262295 | 0.0685868 | -0.0306089 |
| $d_1$ | -1 | -0.262295 | -0.0685868 | 0.0306089 |
| $S_0$ | 0 | 0.737705 | 0.1929 | -0.0860847 |
| $d_2$ | 0 | -1 | -0.261487 | 0.116696 |
| $\alpha$ | 0 | 0.262295 | 0.0685868 | -0.0306089 |
| $r_{10}$ | 0 | 0 | 1.41735 | -0.0429177 |
| $r_{11}$ | 0 | 0 | -0.233076 | 0.00705757 |
| $S_n$ | 0 | 0 | -0.578701 | 0.258262 |
| $c_1$ | 0 | 0 | -1.00542 | 0.0304442 |
| $a_{10}$ | 0 | 0 | -0.118325 | 0.0035829 |
| $a_{11}$ | 0 | 0 | -0.0605356 | 0.00183303 |
| $r_{20}$ | 0 | 0 | -0.160686 | 0.903252 |
| $r_{21}$ | 0 | 0 | -0.0235906 | 0.132608 |
| $c_2$ | 0 | 0 | 0.17886 | -1.00542 |
| $a_{20}$ | 0 | 0 | 0.00542783 | -0.0305111 |
| $a_{21}$ | 0 | 0 | -0.000011901 | 0.0000668984 |

Table 3: Sensitivity indices of the state variables at the equilibrium point $E_{44}$ to the parameters ($x_j$) of the model in the case B where $\gamma$=sensitivity index

increases whereas the equilibrium density of $N_2$ species decreases. From fig.(7) it is noted in the case B ($S_w < S_n$) that as the input rate of pollutant (Q) decreases then the equilibrium density of $N_1$ species decreases whereas the equilibrium density of $N_2$ species increases. In the case A, it is observed that as the depletion rate of pollutant ($d_1$) decreases then equilibrium density of $N_1$ species increases but on the other hand equilibrium density of $N_2$ species decreases (see fig.(8)). While in the case B when the depletion rate of pollutant ($d_1$) increases then the equilibrium density of $N_1$ species decreases whereas the equilibrium density of $N_2$ species increases (see fig.(9)). From fig.(10) it is shown in the case A that as the input rate of salts associated with pollutants inflow in water on account of anthropogenic activities increases then the equilibrium density of $N_1$ species increases whereas the equilibrium density of $N_2$ species decreases. On the other hand in the case B it is seen that as the input rate of salts associated with pollutants inflow in water on account of anthropogenic activities decreases then the equilibrium density of $N_1$ species decreases whereas the equilibrium density of $N_2$ species increases(see fig.(11)). Thus, the results obtained from both analytical and numerical analyses are supported by the results of experimental study [25] in which the authors have observed that the salinity stress altered the competitive balance between Spartina and Phragmites and it is found that the Spartina had the competitive dominance at the higher salinity level whereas Phragmites tended to dominate over Spartina at lower salinity level although Phragmites performance was reduced at this salinity level. Further, the results derived out of analytical and numerical analysis are also supported by the experimental study conducted by Barbour[3] in which the author has observed that as salinity increases then the growth rate of both the species Jaumea and Lolium decrease. The author also observed that the inhibiting effect of competition by Lolium on Jaumea was reduced as salinity level increased.

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