Effect of Pollution on Ratio-Dependent Three-Species Food-Chain System: A Mathematical Approach

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Abstract

In order to investigate the impact of pollution on a three species ratio dependent food chain system, a nonlinear mathematical model is proposed and examined in this research. The primary mathematical model is broken down into submodels in order to study the impact of pollutants. It is assumed in the model that the pollutant has direct adverse impacts on all biological species in the three species food chain system. The models' stability analysis is completed, and the outcomes provide the necessary circumstances for the populations' survival and extinction under the pressure of contaminants. Each possible equilibrium is subjected to a local and global stability study. According to the study, chaotic and limit cycle behaviour may not be identified due to the amount of contaminants present in the biological system. With regard to crucial factors for non-trivial equilibrium locations, a Hopf bifurcation analysis has been conducted. Additionally, numerical simulations are used to support our analytical results.

Keywords: *Food-Chain; Pollutant; Equilibrium; Stability; Hopf-bifurcation*

MSC Classification 93A30; 92D40; 34D20; 34F10

1. Introduction:

Pollutants in the environment have an impact on the biological populations in both terrestrial and aquatic ecosystems, and ecologists continue to face challenging challenges in addressing the issues posed by the presence of pollutants in an ecosystem. Pollutants and toxicants typically slow the growth rate and carrying capacity of biological species. The general goal of ecologists and environmentalists is to prevent the extinction of species and to maintain the diversity of species in the stressed ecosystem.

For studying prey-predator food chains and predicting the survival or extinction of species, mathematical models have recently become essential tools and methodologies [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]. The dynamical study of prey and the animal of prey have long been and will have continued to be the governing topics in the field of ecology due to its popular existence and significance [14, 15, 16].

The dynamical systems embroiled the mathematical modeling of predator prey problems may seem to be elementary; however, the detailed study and analysis of these prey predator systems often initiate convoluted as well as challenging problems. The fundamentals of modeling in population of ecosystem are to reveal the concerned prey predator mathematical model can disclose under certain considerations of well system behavior. The dynamical modeling of prey predator ecological systems is often evolving the procedure. A structured prey predator mathematical model approach can proceed towards a clear apprehension of the feasible lead to the indispensable modifications [16, 17].

A few authors have also used mathematical models to understand the ratio-dependent predator-prey systems [18, 19, 20, 21, 22, 24, 25, 26, 27, 28, 29, 30]. The logistic equation is generally considered to be the prey's growth in these mathematical models of predator and prey that use ratio-dependent functional responses, however other writers have included the idea of the Allee effect in the prey growth function and analysed the ratio-dependent models [20]. In the last few years, a lot of research has been done using mathematical models to examine the dynamic behaviour of tri-trophic level food chains [31, 32, 33, 34]. Toxicants' effects on biological populations in a polluted environment have, however, been the subject of a number of studies over the past ten years, utilising mathematical models [35, 36, 37, 38, 39, 40]. There is a example of the marine considering three species food chain ecosystem of algae zooplankton molluscs, [42] in the research the authors have discussed the food–chain of nonlinear dynamics of algae/phytoplankton toxin emission on the system.

Also there is a another marine example could be observed the tri species food chain ecosystem composed of phytoplankton zooplankton fish, [43] the researchers have learnt that the grazing pressure of zooplankton species reduces due to the toxin producing phytoplankton and also observed from study that the dynamics of food–chain systems perform very less chaotic behaviour.

In order to examine the effects of pollutants, this work proposes and analyses a mathematical model of a food chain system that depends on the ratio of three species. Analytical analysis of the model and numerical simulations of the outcomes are performed.

2. The Mathematical Model:

To examine the impact of toxicant in the ecological food chain model, the system of differential equations listed below is taken into consideration. The model was developed in light of [41] and included the effect of toxicant concentration, which has an immediate impact on the species. The model's food chain model is made up of the following state variables: Prey's density is $X(t)$; intermediate predator species density is $Y(t)$; top predator species density is $Z(t)$; and toxicant concentration $C(t)$. Model makes the assumption that the concentration of toxicants directly affects each species.

The parameter b_i in the model specifies impact of the intra species competition for sustenance. In the logistic growth model, the carrying capacity is denoted by the ratio $a_i/b_i = K_i$. The parameter c_i denotes the influence of an individual at a lower trophic level, whereas d_i denotes the influence of an individual at a higher extent trophic on the species' per capita development (consider i=1, 2, and 3).

The terms $d_1Y/(\gamma_1+X)$ and $d_2Z/(\gamma_2+Y)$ which are Holling type II functional responses—define the existence of alternate prey for the predators. The number e_i represents the species death rate as a result of the unbroken toxicant concentration (consider i=1, 2, and 3). The rate at which the toxicant is thought to wash out is supposed to be α, and the rate at which trophic-level species perish as a result of the toxicant concentration is β. The harmful substance's external input into the environment is Q_0.

The dynamics of the feeding interactions of food chain in a three-species model with a toxicant are controlled by the system of equations:

Model-A:

$$\begin{aligned} \frac{dX}{dT} &= X\left(a_1 - b_1X - \frac{d_1Y}{\gamma_1+X} - e_1C\right) \\ \frac{dY}{dT} &= Y\left(a_2 - b_2Y - \frac{Y}{c_2X} - \frac{d_2Z}{\gamma_2+Y} - e_2C\right) \end{aligned} \tag{1}$$

2 THE MATHEMATICAL MODEL:

$$\begin{aligned}\frac{dZ}{dT} &= Z\left(a_3 - b_3 Z - \frac{Z}{c_3 Y} - e_3 C\right)\\ \frac{dC}{dT} &= Q_0 - \alpha C - \beta(X+Y+Z)C\end{aligned}$$

and $X(0), Y(0), Z(0) > 0$, $C(0) = 0$.
By using the following scaling changes, the number of parameters in the aforementioned system (1) can be decreased:

$$x = \frac{X}{K_1}, \quad y = \frac{d_1 Y}{a_1 K_1}, \quad z = \frac{d_1 d_2 Z}{a_1^2 K_1}, \quad c = \frac{d_1 C}{a_1}, \quad t = a_1 T.$$

Then, the equations in the $Model$-A take the non-dimensionalized form.:
Model 1:

$$\frac{dx}{dt} = x(1-x) - \frac{xy}{u_1 + x} - v_1 xc \tag{2}$$

$$\frac{dy}{dt} = y\left(u_2 - u_3 y - \frac{u_4 y}{x}\right) - \frac{yz}{u_5 + y} - v_2 yc \tag{3}$$

$$\frac{dz}{dt} = z\left(u_6 - u_7 z - \frac{u_8 z}{y}\right) - v_3 zc \tag{4}$$

$$\frac{dc}{dt} = q_0 - \alpha_1 c - (\beta_1 x + \beta_2 y + \beta_3 z)c \tag{5}$$

and $x(0), y(0), z(0) > 0$, $c(0) = 0$.
In this case, the non-dimensional parameters are

$$\begin{array}{cccccc} u_1 = \frac{\gamma_1}{K_1}, & u_2 = \frac{a_2}{a_1}, & u_3 = \frac{b_2 K_1}{d_1}, & u_4 = \frac{1}{d_1 c_2}, & u_5 = \frac{\gamma_2 d_1}{a_1 K_1}, & u_6 = \frac{a_3}{a_1},\\ u_7 = \frac{a_1 K_1 b_3}{d_1 d_2}, & u_8 = \frac{1}{d_2 c_3}, & v_1 = \frac{e_1}{d_1}, & v_2 = \frac{e_2}{d_1}, & v_3 = \frac{e_3}{d_1}, & q_0 = \frac{Q_0 d_1}{a_1^2},\\ & \alpha_1 = \frac{\alpha}{a_1}, & \beta_1 = \frac{\beta K_1}{a_1}, & \beta_2 = \frac{\beta K_1}{d_1}, & \beta_3 = \frac{a_1 K_1 \beta}{d_1 d_2}. & \end{array}$$

By simply taking into account intraspecies competitions for the top prey, $Model$ 1 is made simpler $i.e.$, $u_3 = u_7 = 0$ ($b_2 = b_3 = 0$ in $Model$ A). Additional parameter reductions are made, and Model 1 uses non-dimensionalized equations:
Model 2:

$$\frac{dx}{dt} = x(1-x) - \frac{xy}{u_1 + x} - v_1 xc \tag{6}$$

$$\frac{dy}{dt} = y\left(u_2 - \frac{u_4 y}{x}\right) - \frac{yz}{u_5 + y} - v_2 yc \tag{7}$$

$$\frac{dz}{dt} = z\left(u_6 - \frac{u_8 z}{y}\right) - v_3 zc \tag{8}$$

$$\frac{dc}{dt} = q_0 - \alpha_1 c - (\beta_1 x + \beta_2 y + \beta_3 z)c \tag{9}$$

and $x(0), y(0), z(0) > 0$, $c(0) = 0$.

[41] To investigate the interacting system's long-term dynamic dynamics (6)–(9), two separate systems make up the system.
Subsystem 1 of Model 2: The subsystem 1 is obtained by assuming the absence of top predator population:

$$\frac{dx}{dt} = x(1-x) - \frac{xy}{u_1 + x} - v_1 xc \tag{10}$$

$$\frac{dy}{dt} = y\left(u_2 - \frac{u_4 y}{x}\right) - v_2 yc \tag{11}$$

$$\frac{dc}{dt} = q_0 - \alpha_1 c - (\beta_1 x + \beta_2 y)c \tag{12}$$

Subsystem 2 of Model 2: When there is non-trivial equilibrium in the prey population, the subsystem 2 is obtained ($x = \bar{\bar{x}}$):

$$\frac{dy}{dt} = y\left(u_2 - \frac{u_4 y}{\bar{\bar{x}}}\right) - \frac{yz}{u_5 + y} - v_2 yc \quad (13)$$

$$\frac{dz}{dt} = z\left(u_6 - \frac{u_8 z}{y}\right) - v_3 zc \quad (14)$$

$$\frac{dc}{dt} = q_0 - \alpha_1 c - (\beta_1 \bar{\bar{x}} + \beta_2 y + \beta_3 z)c \quad (15)$$

Numerical simulations are used to discuss and examine the dynamic behaviour of these two subsystems [41].

3. Analysis of Model 1

3.1. Boundedness of the Model 1

In this section, we will evaluate $Model$ 1 and require the boundaries of the related dependent variables. Therefore, the following lemma contains the region of interest for the $Model$ 1.
Lemma: The set

$$\Omega_1 = \{(x, y, z, c) \in R^4_+ : 0 \le x(t) \le 1, 0 \le y(t) \le y_M, 0 \le z(t) \le z_M, \\ 0 \le c(t) \le c_M, W_1(t) \ge W_L\}$$

where $\phi_1 = max\{1 + u_2/u_3 + q_0 v_1/\alpha_1, (u_3 + u_4)u_2/u_3 + u_6/u_7 + q_0 v_2/\alpha_1, (u_7 + u_8)u_6/u_7 + q_0 v_3/\alpha_1, \alpha_1 + \beta_1 + \beta_2 u_2/u_3 + \beta_3 u_6/u_7\}$, is an area where solutions that start in the positive region's interior are drawn to.
Proof: Starting with (2) we get, $dx/dt \le x(1 - x)$ then using comparison theorem, we obtain as $t \to \infty$,

$$x \le 1$$

Let's take a look at the result of the equation (3), now

$$\frac{dy}{dt} \le y(u_2 - u_3 y)$$

then, using the standard comparison theorem, we arrive at $t \to \infty$,

$$y \le \frac{u_2}{u_3} = y_M.$$

Let's take a look at the result of the equation (4), now

$$\frac{dz}{dt} \le z(u_6 - u_7 z)$$

as a result, using the standard comparison theorem, we obtain at $t \to \infty$,

$$z \le \frac{u_6}{u_7} = z_M.$$

Let's take a look at the result of the equation (5), now

$$\frac{dc}{dt} + \alpha_1 c \le q_0$$

as a result, using the standard comparison theorem, we obtain at $t \to \infty$,

$$c \le \frac{q_0}{\alpha_1} = c_M.$$

Let's explore the subsequent function now:

$$W_1(t) = x(t) + y(t) + z(t) + c(t)$$

by using equations (2) to (5), we get

$$\frac{dW_1}{dt} + \phi_1 W_1 \geq q_0$$

where $\phi_1 = max\{1 + u_2/u_3 + q_0v_1/\alpha_1, (u_3 + u_4)u_2/u_3 + u_6/u_7 + q_0v_2/\alpha_1, (u_7 + u_8)u_6/u_7 + q_0v_3/\alpha_1, \alpha_1 + \beta_1 + \beta_2u_2/u_3 + \beta_3u_6/u_7\}$ as a result, using the standard comparison theorem, we obtain at $t \to \infty$,

$$W_1 \geq \frac{q_0}{\phi_1} = W_L$$

Therefore, the $Model$ 1 solution is bounded in Ω_1.

3.2. Equilibria of Model 1

There are four positive equilibria in the "Model 1" x, y, z and c, $\hat{E_{10}}(0,0,0,\hat{c})$, $\ddot{E_{11}}(\ddot{x},0,0,\ddot{c})$, $\tilde{E_{12}}(\tilde{x},\tilde{y},0,\tilde{c})$ & $\bar{E_{13}}(\bar{x},\bar{y},\bar{z},\bar{c})$. Let's establish the validity of $\hat{E_{10}}$, $\ddot{E_{11}}$, $\tilde{E_{12}}$ and $\bar{E_{13}}$ as follows:

- **The existence at** $\hat{E_{10}} = (0,0,0,\hat{c})$
  from (5),

$$\hat{c} = \frac{q_0}{\alpha_1} \tag{16}$$

- **The existence at** $\ddot{E_{11}}(\ddot{x},0,0,\ddot{c})$,
  begin with (2),

$$\ddot{x} = 1 - v_1\ddot{c} \tag{17}$$

  $\ddot{x} > 0$ if $1 > v_1\ddot{c}$,
  from (5) we get

$$\beta_1 v_1 \ddot{c}^2 - (\alpha_1 + \beta_1)\ddot{c} + q_0 = 0 \tag{18}$$

  then the positive root of (18) is

$$\ddot{c} = \frac{(\alpha_1 + \beta_1) + \sqrt{(\alpha_1 + \beta_1)^2 - 4q_0\beta_1v_1}}{2\beta_1v_1} > 0$$

  $\ddot{c} > 0$ if $(\alpha_1 + \beta_1)^2 > 4q_0\beta_1v_1$, and $\alpha_1 < q_0v_1$.

- **The existence at** $\tilde{E_{12}}(\tilde{x},\tilde{y},0,\tilde{c})$,
  considering (3) and (2),

$$y = \frac{x(u_2 - \frac{v_2}{v_1} + \frac{v_2}{v_1}x)}{(xu_3 + u_4 - \frac{v_2}{v_1}\frac{x}{u_1+x})} = f_1(x) \tag{19}$$

  from (2),

$$c = \frac{1}{v_1}\left(1 - x - \frac{f_1(x)}{u_1 + x}\right) = f_2(x) \tag{20}$$

  Let

$$F(x) = q_0 - \alpha_1 f_2(x) - (\beta_1 x + \beta_2 f_1(x))f_2(x) \tag{21}$$

  Then we note that

$$F(0) = (q_0v_1 - \alpha_1)/v_1 > 0 \tag{22}$$

  if $q_0v_1 > \alpha_1$ and

$$F(k_0) = q_0 - \alpha_1 f_2(k_0) - (\beta_1 k_0 + \beta_2 f_1(k_0))f_2(k_0) < 0 \tag{23}$$

  The presence of a root is therefore assured $F(x) = 0$ for $0 < x < k_0$, suppose $\tilde{x}$. Additionally, this root will be distinct if

$$F'(x) = -\alpha_1 f_2'(x) - [(\beta_1 x + \beta_2 f_1(x))f_2'(x) + f_2(x)(\beta_1 + \beta_2 f_1'(x))] < 0 \tag{24}$$

  Understanding the importance of $\tilde{x}$, then the values of $\tilde{y}$ and $\tilde{c}$ is calculable from equations (19) and (20) respectively.

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- **The existence at** $\bar{E_{13}}(\bar{x}, \bar{y}, \bar{z}, \bar{c})$,
  from (2),

$$c = \frac{1}{v_1}\left(1 - x - \frac{y}{u_1 + x}\right) = g_1(x, y) \tag{25}$$

from (5),

$$z = \frac{1}{\beta_3 g_1(x, y)}\left(q_0 - \alpha_1 c - (\beta_1 x + \beta_2 y)c\right) = g_2(x, y) \tag{26}$$

Now, have a look at two functions,

$$\begin{aligned} G_{11}(x, y) &= (1 - x)(u_1 + x) - y - v_1 c(u_1 + x) + (u_6 - u_7 z - v_3 c) \\ &+ (u_5 + y)(x(u_2 - u_3 y) - u_4 y - v_2 c x) - (u_8 + x)z = 0 \quad (27) \\ G_{12}(x, y) &= q_0 - \alpha_1 c - (\beta_1 x + \beta_2 y + \beta_3 z)c \\ &+ y(u_6 - u_7 z - v_3 c) - u_8 z = 0 \quad (28) \end{aligned}$$

For the existence at $\bar{x}$ and $\bar{y}$, then two isoclines,

$$G_{11}(x, y) = 0 \tag{29}$$

$$G_{12}(x, y) = 0 \tag{30}$$

should cross.
The fact that

$$G_{11}(0, 0) = G_{12}(0, 0) = \frac{u_8}{\beta_3}(\alpha_1 - q_0 v_1) > 0$$

if $\alpha_1 > q_0 v_1$. Also,
$G_{11}(0, y) = 0$ then y a single positive root (ψ_1 suppose), is determined by the cubic equation of y,

$$D_{11} y^3 + D_{12} y^2 + D_{13} y - D_{14} = 0$$

where
$D_{11} = \beta_3(u_4 + \beta_2 u_7) - v_3\beta_3/v_1 u_1$,
$D_{12} = \alpha_1 u_7 + \beta_3 u_4 u_5 + u_8\beta_2\beta_3 + 2v_3\beta_3/v_1 - \beta_3(u_1 u_4 + u_6 + \beta_2 u_1 u_7)$,
$D_{13} = u_1\beta_3(1 + u_6 - u_4 u_5) - u_1 v_3\beta_3/v_1 + v_1 u_1 q_0 u_7 + \alpha_1 u_8 - \alpha_1 u_1 u_7 - \beta_2\beta_3 u_1 u_8$,
$D_{14} = u_1(\alpha_1 u_8 + u_1\beta_3) - q_0 v_1 u_1 u_8$.
Here, $D_{11} > 0$, $D_{12} > 0$, $D_{13} > 0$ and $D_{14} > 0$.
$G_{11}(x, 0) = 0$ then x a single positive root (ψ_2 suppose), is determined by the cubic equation of x,

$$D_{15} x^3 + D_{16} x^2 + D_{17} x - D_{18} = 0$$

where
$D_{15} = \beta_1 + \beta_3 v_2 u_5/v_1$,
$D_{16} = \alpha_1 + \beta_1 u_8 + u_2 u_5\beta_3 - 2v_2\beta_3 u_5/v_1$,
$D_{17} = q_0 v_1 - \alpha_1 + \alpha_1 u_8 - \beta_1(1 + u_8) + v_2\beta_3 u_5/v_1 - u_2 u_5\beta_3$,
$D_{18} = u_8(\alpha_1 - q_0 v_1)$.
Here, $D_{15} > 0$, $D_{16} > 0$, $D_{17} > 0$ and $D_{18} > 0$.
$G_{12}(0, y) = 0$ then y a single positive root (ψ_3 say), is determined by the cubic equation of y,

$$D_{21} y^3 + D_{22} y^2 + D_{23} y - D_{24} = 0$$

where
$D_{21} = (\beta_2 + v_3)/v_1 u_1^2 + \beta_2(u_7 - \beta_3/v_1 u_1)/(u_1\beta_3)$,
$D_{22} = u_6/u_1 + \beta_2(u_8 + \beta_3/v_1)/(u_1\beta_3) + (u_7 - \beta_3/(v_1 u_1))(\alpha_1/u_1 - \beta_2)/\beta_3 - (\beta_2 + v_3)/(v_1 u_1) - (\beta_2 + v_3 - \alpha_1/u_1)/(v_1 u_1)$,
$D_{23} = (u_8 + \beta_3/v_1)(\alpha_1/u_1 - \beta_2)/\beta_3 - (u_7 - \beta_3/v_1 u_1)(\alpha_1 - v_1 q_0)/\beta_3 + q_0/u_1 + (\beta_2 + v_3 - \alpha_1/u_1)/v_1 - u_6 - \alpha_1/v_1 u_1$,

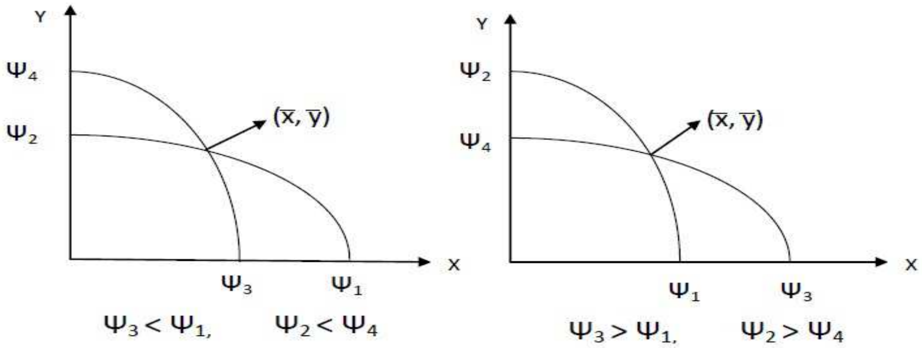

Fig.4.1

$D_{24} = (u_8 + \beta_3/v_1)(\alpha_1 - v_1 q_0)/\beta_3 - (\alpha_1 - v_1 q_0)/v_1$.
Here, $D_{21} > 0$, $D_{22} > 0$, $D_{23} > 0$ and $D_{24} > 0$.
$G_{12}(x, 0) = 0$ then $x = \sqrt{\frac{(\alpha_1 - v_1 q_0)}{\beta_1}}$ if $\alpha_1 > v_1 q_0$ then the root (ψ_4 suppose).
Consequently, two isoclines cross each other in the area

$$M = \{(x, y) : 0 < x < \psi_4, 0 < y < \psi_1\}$$

in the two following situations: (see Fig.4.1)

$$Case(i) : \psi_1 > \psi_3, \psi_4 > \psi_2 \tag{31}$$

$$Case(ii) : \psi_1 < \psi_3, \psi_4 < \psi_2 \tag{32}$$

This junction will result in $\bar{x}$, $\bar{y}$. For distinctiveness of $(\bar{x}, \bar{y})$, we must have $\frac{dy}{dx} < 0$ regarding both of the region's curves M.
For curve (29),

$$\frac{dy}{dx} = \frac{\Phi_{12} - (v_1 c + z) + \frac{\Phi_{11}}{v_1}(1 - \frac{y}{(u_1+x)^2})}{\Phi_{14} + 1 - (u_6 - u_7 z - v_3 c) - \Phi_{13} - \frac{\Phi_{11}}{v_1(u_1+x)}} < 0 \tag{33}$$

and for curve (30),

$$\frac{dy}{dx} = \frac{\frac{\Phi_{21}}{v_1}(1 - \frac{y}{(u_1+x)^2}) - \beta_1 c}{\beta_2 c - (u_6 - u_7 z - v_3 c) - \frac{\Phi_{21}}{v_1(u_1+x)}} < 0 \tag{34}$$

where,
$\Phi_{11} = v_1(u_1 + x) + (v_2 x + v_3 y) - (x + u_7 y + u_8)\frac{q_0}{c^2 \beta_3}$,
$\Phi_{12} = (y + u_5)(u_2 - u_3 y - v_2 c) + (1 - u_1 - 2x)$,
$\Phi_{13} = x(u_2 - u_3 y) - (u_4 y + v_2 c x)$,
$\Phi_{14} = (y + u_5)(u_4 + x u_3)$,
$\Phi_{21} = (\alpha_1 + x\beta_1 + y(\beta_2 + v_3) + z\beta_3) - (c\beta_3 + u_7 y + u_8)\frac{q_0}{c^2 \beta_3}$.
In case (i), the exact value of $\frac{dy}{dx}$ given by (33) is lower than the absolute value of $\frac{dy}{dx}$ given by (34). For the case (ii), the situation is the exact opposite.
Having knowledge of $\bar{x}$, $\bar{y}$; $\bar{z}$ and $\bar{c}$ is calculable from the equations (25)-(26).

3.3. Stability of Equilibria of Model 1

The eigenvalues of the variational matrix surrounding each equilibrium are calculated in order to ascertain the local stability of each equilibrium in $Model$ 1.

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According to $Model1$, the general variational matrix is

$$J_1 = \begin{bmatrix} c_{11} & -\frac{x}{x+u_1} & 0 & -xv_1 \\ \frac{u_4 y^2}{x^2} & c_{22} & -\frac{y}{y+u_5} & -yv_2 \\ 0 & \frac{u_8 z^2}{y^2} & -z(u_7+\frac{u_8}{y})+f_{33} & -zv_3 \\ -\beta_1 c & -\beta_2 c & -\beta_3 c & -(\alpha_1+\beta_1 x+\beta_2 y+\beta_3 z) \end{bmatrix}$$

where, $c_{11} = -x(1-\frac{y}{(u_1+x)^2})+f_{11}$, $c_{22} = -y(u_3+\frac{u_4}{x}-\frac{z}{(u_5+y)^2})+f_{22}$, $f_{11} = 1-x-\frac{y}{u_1+x}-v_1 c$, $f_{22} = u_2-u_3 y-\frac{u_4 y}{x}-\frac{z}{u_5+y}-v_2 c$, $f_{33} = u_6-u_7 z-\frac{u_8 z}{y}-v_3 c$.
In light of this, the examination of linear stability regarding the equilibrium points $\ddot{E_{11}}, \tilde{E_{12}}$ and $\bar{E_{13}}$ produces the following outcomes:

- **At the point** $\ddot{E_{11}}$, in the characteristic equation, there are two eigenvalues: $u_2 - v_2\ddot{c}$ and $u_6 - v_3\ddot{c}$ and the roots of the following equation yield the other two eigenvalues:

$$\lambda^2 + \lambda K_{11} + K_{12} = 0 \tag{35}$$

where, $K_{11} = \alpha_1 + (1+\beta_1)(1-v_1\ddot{c}) > 0$, $K_{12} = (1-v_1\ddot{c})[\alpha_1+\beta_1-2\beta_1 v_1\ddot{c}] > 0$.
Equation using the Routh-Hurwitz criterion (35), it has been noted that $\ddot{E_{11}}$ assuming that is locally asymptotically stable K_{11} and K_{12} are positive, *i.e.*, $(\alpha_1+\beta_1\ddot{x}) > \beta_1 v_1\ddot{c}$, $\ddot{c} > \frac{u_2}{v_2}$, $\ddot{c} > \frac{u_6}{v_3}$ and $2\alpha < \beta K_1$.
Remark 1: According to the stability and from its conditions of the equilibrium $\ddot{E_{11}}$, the following biological phenomenon is noted to occur when the twice of wash out rate of toxic is less than the multiply of the death rate of the species in the three level chain due to the toxicant concentration and the carrying capacity of the species: the prey population will survive and the predator population in the food chain will tend to/go extinct.

- **At the point** $\tilde{E_{12}}$, we observe that one of the characteristic's eigenvalues is $u_6 - v_3\tilde{c}$, *i.e.*,

$$\tilde{c} > \frac{u_6}{v_3}$$

where the other three eigenvalues are determined by the following equation's roots:

$$\lambda^3 + \lambda^2 Q_{11} + \lambda Q_{12} + Q_{13} = 0 \tag{36}$$

where,
$Q_{11} = (\alpha_1+\beta_1\tilde{x}+\beta_2\tilde{y}) + \tilde{x}(1-\frac{\tilde{y}}{(u_1+\tilde{x})^2}) + \tilde{y}(u_3+\frac{u_4}{\tilde{x}}) > 0$,

$Q_{12} = \frac{u_4\tilde{y}^2}{\tilde{x}(u_1+\tilde{x})} + \tilde{x}\tilde{y}(1-\frac{\tilde{y}}{(u_1+\tilde{x})^2})(u_3+\frac{u_4}{\tilde{x}}) + (\alpha_1+\beta_1\tilde{x}+\beta_2\tilde{y})(\tilde{x}(1-\frac{\tilde{y}}{(u_1+\tilde{x})^2})+\tilde{y}(u_3+\frac{u_4}{\tilde{x}})) - \tilde{x}\tilde{c}v_1\beta_1 - \tilde{y}\tilde{c}v_2\beta_2 > 0$,

$Q_{13} = (\frac{u_4\tilde{y}^2}{\tilde{x}(u_1+\tilde{x})} + \tilde{x}\tilde{y}(1-\frac{\tilde{y}}{(u_1+\tilde{x})^2})(u_3+\frac{u_4}{\tilde{x}}))(\alpha_1+\beta_1\tilde{x}+\beta_2\tilde{y}) + \tilde{y}\tilde{c}v_2\beta_1\frac{\tilde{x}}{u_1+\tilde{x}} - \tilde{x}\tilde{y}\tilde{c}v_1\beta_1(u_3+\frac{u_4}{\tilde{x}}) - \tilde{c}v_1\beta_2\frac{u_4\tilde{y}^2}{\tilde{x}} - \tilde{x}\tilde{y}\tilde{c}v_2\beta_2(1-\frac{\tilde{y}}{(u_1+\tilde{x})^2}) > 0$.

Equation using the Routh-Hurwitz criterion (36), it has been noted that $\tilde{E_{12}}$ is locally asymptotically stable provided that Q_{11}, Q_{12} and Q_{13} are positive and $Q_{11}Q_{12} > Q_{13}$ *i.e.*,
$(\alpha_1+\beta_1\tilde{x}+\beta_2\tilde{y}+\tilde{y}(u_3+\frac{u_4}{\tilde{x}}))(\tilde{x}^2(1-\frac{\tilde{y}}{(u_1+\tilde{x})^2})^2 - \tilde{y}\tilde{c}v_2\beta_2 + \tilde{x}(1-\frac{\tilde{y}}{(u_1+\tilde{x})^2})(\alpha_1+\beta_1\tilde{x}+\beta_2\tilde{y}-\tilde{y}(u_3+\frac{u_4}{\tilde{x}})) + \tilde{y}(\alpha_1+\beta_1\tilde{x}+\beta_2\tilde{y})(u_3+\frac{u_4}{\tilde{x}})) + \frac{u_4\tilde{y}^2}{\tilde{x}(u_1+\tilde{x})}(\tilde{x}(1-\frac{\tilde{y}}{(u_1+\tilde{x})^2})+\tilde{y}(u_3+\frac{u_4}{\tilde{x}})) - \tilde{x}\tilde{c}v_1\beta_1(\tilde{x}(1-\frac{\tilde{y}}{(u_1+\tilde{x})^2}) + \alpha_1+\beta_1\tilde{x}+\beta_2\tilde{y}) + 2\tilde{x}\tilde{y}(1-\frac{\tilde{y}}{(u_1+\tilde{x})^2})(\alpha_1+\beta_1\tilde{x}+\beta_2\tilde{y})(u_3+\frac{u_4}{\tilde{x}}) + \tilde{c}v_1\beta_2\frac{u_4\tilde{y}^2}{\tilde{x}} - \tilde{y}\tilde{c}v_2\beta_1\frac{\tilde{x}}{u_1+\tilde{x}} > 0$.

- The equilibrium point's characteristic equation, $\bar{E_{13}}$, is stated as follows:

$$\lambda^4 + \lambda^3 U_1 + \lambda^2 U_2 + \lambda U_3 + U_4 = 0 \tag{37}$$

where,
$U_1 = m_1 + m_4 + m_7 + m_8$,

$U_2 = m_1m_8 + (m_1 + m_8)(m_4 + m_7) + m_2m_3 + m_5m_6 + m_4m_7 - \bar{c}(\bar{x}v_1\beta_1 + \bar{y}v_2\beta_2 + \bar{z}v_3\beta_3)$,

$U_3 = (m_1 + m_8)(m_5m_6 + m_4m_7) + (m_4 + m_7)(m_1m_8 - \bar{x}\bar{c}v_1\beta_1) - (m_1 + m_7)\bar{y}\bar{c}v_2\beta_2 - (m_1 + m_4)\bar{z}\bar{c}v_3\beta_3 + \bar{y}\bar{c}m_2v_2\beta_1 - \bar{x}\bar{c}v_1m_3\beta_2 + \bar{z}\bar{c}m_5v_3\beta_2 - \bar{y}\bar{c}v_2m_6\beta_3$,

$U_4 = (m_1m_4 + m_2m_3)(m_7m_8 - \bar{z}\bar{c}v_3\beta_3) - \bar{c}(m_7\beta_2 + m_6\beta_3)(\bar{x}v_1m_3 + \bar{y}m_1v_2) + m_1m_5(\bar{z}\bar{c}v_3\beta_2 + m_6m_8) + \bar{c}m_2\beta_1(\bar{y}v_2m_7 - m_5m_7) - \bar{x}\bar{c}v_1\beta_1(m_5m_6 + m_4m_7)$

and
$m_1 = \bar{x}(1 - \frac{\bar{y}}{(u_1+\bar{x})^2}), m_2 = \frac{\bar{x}}{\bar{x}+u_1}, m_3 = \frac{u_4\bar{y}^2}{\bar{x}^2}, m_4 = \bar{y}(u_3 + \frac{u_4}{\bar{x}} - \frac{\bar{z}}{(u_5+\bar{y})^2}), m_5 = \frac{\bar{y}}{\bar{y}+u_5}, m_6 = \frac{u_8\bar{z}^2}{\bar{y}^2}, m_7 = \bar{z}(u_7 + \frac{u_8}{\bar{y}}), m_8 = \alpha_1 + \beta_1\bar{x} + \beta_2\bar{y} + \beta_3\bar{z}$.
The Routh-Hurwitz criterion states that if certain conditions are met, then $\bar{E_{13}}$ is locally asymptotically stable:

U_1, U_2, U_3 and U_4 and $U_1U_2U_3 > U_3^2 + U_1^2U_4$ are positive.

3.3.1. Global Stability

The coexisting equilibrium point $\bar{E_{13}}$ global stability will be demonstrated in this section. Using the appropriate Lyapunov function, we demonstrate the global stability result. All paths finally approach the equilibrium point starting from any location within the positive octant because of the coexisting equilibrium's global stability.
Theorem 4.1:
If the area has any of the following disparities in Ω_1,

$$\bar{y} < \sigma_{11} \quad (38)$$

$$\frac{W_L}{\sigma_{13}} < (u_3 + \frac{u_4\bar{x}}{\sigma_{12}}) \quad (39)$$

$$6[\frac{u_1+\bar{x}}{\sigma_{11}} - \frac{u_4\bar{y}}{\sigma_{12}}]^2 < (1 - \frac{\bar{y}}{\sigma_{11}})(u_3 + \frac{u_4\bar{x}}{\sigma_{12}} - \frac{W_L}{\sigma_{13}}) \quad (40)$$

$$6[v_1 + \beta_1 W_L]^2 < (1 - \frac{\bar{y}}{\sigma_{11}})(\alpha_1 + \beta_1\bar{x} + \beta_2\bar{y} + \beta_3\bar{z}) \quad (41)$$

$$6[\frac{1}{\sigma_{13}}(\bar{y} + u_5) - \frac{u_8}{\sigma_{14}}]^2 < (u_3 + \frac{u_4\bar{x}}{\sigma_{12}} - \frac{W_L}{\sigma_{13}})(u_7 + \frac{u_8\bar{y}}{\sigma_{14}}) \quad (42)$$

$$9[v_2 + \beta_2 W_L]^2 < (u_3 + \frac{u_4\bar{x}}{\sigma_{12}} - \frac{W_L}{\sigma_{13}})(\alpha_1 + \beta_1\bar{x} + \beta_2\bar{y} + \beta_3\bar{z}) \quad (43)$$

$$6(v_3 + \beta_3 W_L)^2 < (u_7 + \frac{u_8\bar{y}}{\sigma_{14}})(\alpha_1 + \beta_1\bar{x} + \beta_2\bar{y} + \beta_3\bar{z}) \quad (44)$$

where, $\sigma_{11} = (u_1 + \bar{x})(u_1 + 1)$, $\sigma_{12} = \bar{x}$, $\sigma_{13} = (u_5 + \bar{y})(u_5 + y_M)$ and $\sigma_{14} = y_M\bar{y}$, then all the solutions starting in the interior of the positive region have a global asymptotic stability with regard to the positive equilibrium $\bar{E_{13}}$ in Ω_1.
Proof: The following positive definite function is taken into consideration $\bar{E_{13}}$:

$$V_{11} = (x - \bar{x} - \bar{x}ln(\frac{x}{\bar{x}})) + (y - \bar{y} - \bar{y}ln(\frac{y}{\bar{y}})) + (z - \bar{z} - \bar{z}ln(\frac{z}{\bar{z}})) + \frac{1}{2}(c - \bar{c})^2$$

Differentiate V_{11} with respect to time t, we get

$$\frac{dV_{11}}{dt} = (\frac{x - \bar{x}}{x})\frac{dx}{dt} + (\frac{y - \bar{y}}{y})\frac{dy}{dt} + (\frac{z - \bar{z}}{z})\frac{dz}{dt} + (c - \bar{c})\frac{dc}{dt}$$

Using the (2)-(5) system of equations, we obtain the result after performing several algebraic operations

$$\frac{dV_{11}}{dt} = -(x - \bar{x})^2(1 - \frac{\bar{y}}{\sigma_{11}}) - (y - \bar{y})^2(u_3 + \frac{u_4\bar{x}}{\sigma_{12}} - \frac{z}{\sigma_{13}})$$
$$-(z - \bar{z})^2(u_7 + \frac{u_8\bar{x}}{\sigma_{14}}) - (c - \bar{c})^2(\alpha_1 + \beta_1\bar{x} + \beta_2\bar{y} + \beta_3\bar{z})$$

$$-(x-\bar{x})(y-\bar{y})(\frac{u_1+\bar{x}}{\sigma_{11}}-\frac{u_4\bar{y}}{\sigma_{12}})-(x-\bar{x})(c-\bar{c})(v_1+\beta_1 c)$$
$$-(y-\bar{y})(z-\bar{z})(\frac{1}{\sigma_{13}}(\bar{y}+u_5)-\frac{u_8}{\sigma_{14}})$$
$$-(y-\bar{y})(c-\bar{c})(v_2+\beta_2 c)-(z-\bar{z})(c-\bar{c})(v_3+\beta_3 c)$$

Where, $\sigma_{11}=(u_1+\bar{x})(u_1+x)$, $\sigma_{12}=x\bar{x}$, $\sigma_{13}=(u_5+\bar{y})(u_5+y)$ and $\sigma_{14}=y\bar{y}$.
Now $\frac{dV_{11}}{dt}$ additionally be expressed as the sum of the quadratic forms as

$$\begin{aligned}\frac{dV_{11}}{dt} \leq & -[(\frac{a_{11}}{2}(x-\bar{x})^2+a_{12}(x-\bar{x})(y-\bar{y})+\frac{a_{22}}{2}(y-\bar{y})^2)\\ &+(\frac{a_{11}}{2}(x-\bar{x})^2+a_{14}(x-\bar{x})(c-\bar{c})+\frac{a_{44}}{2}(c-\bar{c})^2)\\ &+(\frac{a_{22}}{2}(y-\bar{y})^2+a_{23}(y-\bar{y})(z-\bar{z})+\frac{a_{33}}{2}(z-\bar{z})^2)\\ &+(\frac{a_{22}}{2}(y-\bar{y})^2+a_{24}(y-\bar{y})(c-\bar{c})+\frac{a_{44}}{2}(c-\bar{c})^2)\\ &+(\frac{a_{33}}{2}(z-\bar{z})^2+a_{34}(z-\bar{z})(c-\bar{c})+\frac{a_{44}}{2}(c-\bar{c})^2)]\end{aligned}$$

where,
$a_{11}=(1-\frac{\bar{y}}{\sigma_{11}})/2$, $a_{12}=\frac{u_1+\bar{x}}{\sigma_{11}}-\frac{u_4\bar{y}}{\sigma_{12}}$, $a_{14}=v_1+\beta_1 c$, $a_{22}=(u_3+\frac{u_4\bar{x}}{\sigma_{12}}-\frac{z}{\sigma_{13}})/3$, $a_{23}=\frac{1}{\sigma_{13}}(\bar{y}+u_5)-\frac{u_8}{\sigma_{14}}$, $a_{24}=v_2+\beta_2 c$, $a_{33}=(u_7+\frac{u_8\bar{y}}{\sigma_{14}})/2$, $a_{34}=v_3+\beta_3 c$, $a_{44}=(\alpha_1+\beta_1\bar{x}+\beta_2\bar{y}+\beta_3\bar{z})/3$.
Using Sylvester's criteria, we now obtain $\frac{dV_{11}}{dt}$ is unambiguously negative under the following circumstances:

$$a_{11} > 0 \quad (45)$$
$$a_{22} > 0 \quad (46)$$
$$a_{11}a_{22} > a_{12}^2 \quad (47)$$
$$a_{11}a_{44} > a_{14}^2 \quad (48)$$
$$a_{22}a_{33} > a_{23}^2 \quad (49)$$
$$a_{22}a_{44} > a_{24}^2 \quad (50)$$
$$a_{33}a_{44} > a_{34}^2 \quad (51)$$

We note that the inequalities, (38) ⇒ (45), (39) ⇒ (46), (40) ⇒ (47), (41) ⇒ (48), (42) ⇒ (49), (43) ⇒ (50) and (44) ⇒ (51).
Hence V_{11} is a Lyapunov demonstrates the theorem with respect to $\bar{E_{13}}$, this domain encompasses the zone of attraction Ω_1.

4. Analysis of Model 2

4.1. Boundedness of the Model 2

In this part, the boundaries of the relevant dependent variables are needed to examine Model 2. Thus, the following lemma contains the region that the Model 2 finds appealing.
Lemma: The set

$$\Omega_2=\{(x,y,z,c)\in R_+^4: \quad 0\leq x(t)\leq 1, 0\leq y(t)\leq y_m, 0\leq z(t)\leq z_m, \\ 0\leq c(t)\leq c_m, x(t)+y(t)+z(t)+c(t)\geq W_l\}$$

where $\phi_2=max\{1+\frac{u_2}{u_4}+\frac{q_0 v_1}{\alpha_1}, u_2+\frac{q_0 v_2}{\alpha_1}+\frac{u_2 u_6}{u_4 u_8}, \frac{u_2 u_6}{u_4}+\frac{q_0 v_3}{\alpha_1}, \alpha_1+\beta_1+\frac{u_2\beta_2}{u_4}+\frac{u_2 u_6\beta_3}{u_4 u_8}\}$, is a region that solutions that start in the interior of the positive zone are drawn to.
Proof: For the interior of the positive region, let us do the following steps:
Equation (6) we obtain, $dx/dt\leq x(1-x)$ next, using the standard comparison theorem, we obtain as $t\to\infty$,

$$x\leq 1$$

Next, let's look at the equation (7), we obtain

$$\frac{dy}{dt} \leq y(u_2 - u_4 y)$$

next, using the standard comparison theorem, we obtain as $t \to \infty$,

$$y \leq \frac{u_2}{u_4} = y_m.$$

Now, again let us consider from (8), we get

$$\frac{dz}{dt} \leq z(u_6 - \frac{u_8}{y} z)$$

next, using the standard comparison theorem, we obtain as $t \to \infty$,

$$z \leq \frac{u_2 u_6}{u_4 u_8} = z_m.$$

Next, let's look at the equation (9), we get

$$\frac{dc}{dt} + \alpha_1 \leq q_0$$

next, using the standard comparison theorem, we obtain as $t \to \infty$,

$$c \leq \frac{q_0}{\alpha_1} = c_m.$$

Next, let's look at the function:

$$W_2(t) = x(t) + y(t) + z(t) + c(t)$$

by using (6) to (9), we get

$$\frac{dW_2}{dt} + \phi_2 W_2 \geq q_0$$

where $\phi_2 = max\{1 + \frac{u_2}{u_4} + \frac{q_0 v_1}{\alpha_1}, u_2 + \frac{q_0 v_2}{\alpha_1} + \frac{u_2 u_6}{u_4 u_8}, \frac{u_2 u_6}{u_4} + \frac{q_0 v_3}{\alpha_1}, \alpha_1 + \beta_1 + \frac{u_2 \beta_2}{u_4} + \frac{u_2 u_6 \beta_3}{u_4 u_8}\}$ next, using the standard comparison theorem, we obtain as $t \to \infty$,

$$W_2 \geq \frac{q_0}{\phi_2} = W_l$$

Hence, the answer to the Model 2 is bounded in Ω_2.

4.2. Equilibria of Model 2

The $Model$ 2 has the four non-negative equilibrium conditions listed below in the x, y, z, and c spaces: $\hat{\hat{E}}_{20}(0,0,0,\hat{\hat{c}})$, $\ddot{\ddot{E}}_{21}(\ddot{\ddot{x}},0,0,\ddot{\ddot{c}})$, $\tilde{\tilde{E}}_{22}(\tilde{\tilde{x}},\tilde{\tilde{y}},0,\tilde{\tilde{c}})$ and $\bar{\bar{E}}_{23}(\bar{\bar{x}},\bar{\bar{y}},\bar{\bar{z}},\bar{\bar{c}})$. It is clear that E_{20} exists. We establish the validity of $\ddot{\ddot{E}}_{21}$, $\tilde{\tilde{E}}_{22}$ and $\bar{\bar{E}}_{23}$ as follows:

- **The Existence at** $\hat{\hat{E}}_{20}(0,0,0,\hat{\hat{c}})$,
  from (9),

$$\hat{\hat{c}} = \frac{q_0}{\alpha_1} \tag{52}$$

- **The Existence at** $\ddot{\ddot{E}}_{21}(\ddot{\ddot{x}},0,0,\ddot{\ddot{c}})$,
  from (6),

$$\ddot{\ddot{x}} = 1 - v_1 \ddot{\ddot{c}} \tag{53}$$

  $\ddot{\ddot{x}} > 0$ if $1 > v_1 \ddot{\ddot{c}}$,
  from (9) we get

$$\beta_1 v_1 \ddot{\ddot{c}}^2 - (\alpha_1 + \beta_1)\ddot{\ddot{c}} + q_0 = 0 \tag{54}$$

then the positive root of (54) is

$$\ddot{c} = \frac{(\alpha_1 + \beta_1) + \sqrt{(\alpha_1 + \beta_1)^2 - 4q_0\beta_1 v_1}}{2\beta_1 v_1} > 0$$

$\ddot{c} > 0$ if $(\alpha_1 + \beta_1)^2 > 4q_0\beta_1 v_1$, and $\alpha_1 < q_0 v_1$.

- **The Existence at** $\tilde{\tilde{E}}_{22}(\tilde{\tilde{x}}, \tilde{\tilde{y}}, 0, \tilde{\tilde{c}})$,
  from (6) and (7),

$$y = \frac{x(x + u_1)[v_1 u_2 - v_2(1 - x)]}{v_1 u_4(x + u_1) - x v_2} = h_1(x) \tag{55}$$

from (6),

$$c = \frac{u_4(1 - x)(x + u_1) - x u_2}{v_1 u_4(x + u_1) - x v_2} = h_2(x) \tag{56}$$

Now, let us consider a function,

$$H(x) = q_0 - \alpha_1 h_2(x) - (\beta_1 x + \beta_2 h_1(x))h_2(x) \tag{57}$$

Then we note that
$H(0) = \frac{1}{v_1}(q_0 v_1 - \alpha_1) > 0$ if $q_0 v_1 > \alpha_1$ and
$H(k_0) = q_0 - \alpha_1 h_2(k_0) - (\beta_1 k_0 + \beta_2 h_1(k_0))h_2(k_0) < 0$.
This ensures that there is a root of this $H(x) = 0$ for $0 < x < k_0$, say $\tilde{\tilde{x}}$. Additionally, this root will be distinct if

$$H'(x) = -(\alpha_1 + \beta_1 x + \beta_2 h_1(x))h_2'(x) - (\beta_1 + \beta_2 h_1'(x))h_2(x) < 0$$

Recognizing the value of $\tilde{\tilde{x}}$, the principle values of y and c can be calculated using formulae (55) to (56) respectively.

- **Existence of** $\bar{\bar{E}}_{23}(\bar{\bar{x}}, \bar{\bar{y}}, \bar{\bar{z}}, \bar{\bar{c}})$,
  from (6),

$$y = (1 - x - v_1 c)(x + u_1) = t_1(x, c) \tag{58}$$

from (8),

$$z = \frac{1}{u_8} t_1(x, c)(u_6 - v_3 c) = t_2(x, c) \tag{59}$$

Consider the function from (7) and (8),

$$T_1(x, c) = x t_1 - x u_8 t_2 - \frac{v_3 t_1}{v_2}(x u_2 - u_4 t_1 - \frac{x t_2}{t_1 + u_5}) \tag{60}$$

and from (7) and (9),

$$\begin{aligned} T_2(x, c) \quad &= \quad q_0 - \alpha_1 c - x t_2 - t_1 u_4(t_1 + u_5) + x(u_2 - v_2 c)(t_1 + u_5) \\ &\quad -(\beta_1 x + \beta_2 t_1(x, c) + \beta_3 t_2(x, c))c \end{aligned} \tag{61}$$

Then we note that

$$\begin{aligned} T_1(0, 0) &= \frac{u_1^2 v_3 u_4}{v_2} > 0 \\ T_2(0, 0) &= q_0 - u_1 u_4(u_1 + u_5) > 0 \end{aligned}$$

if $q_0 > u_1 u_4(u_1 + u_5)$.
Also,

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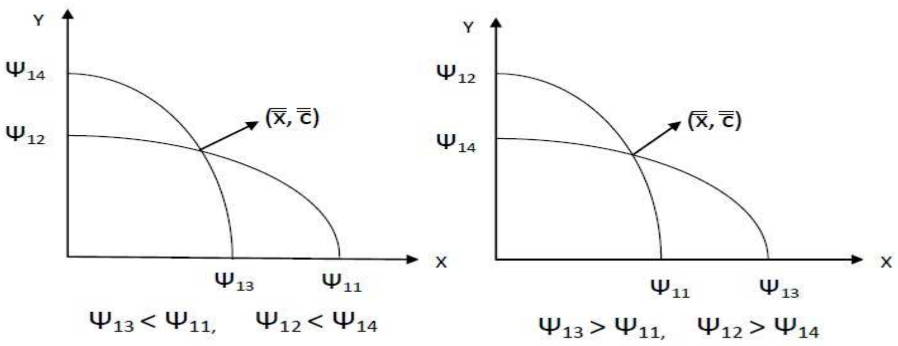

Fig.5.1

$T_1(0, c) = 0$ then $c = 1/v_1$ (ψ_{11} say),

$T_1(x, 0) = 0$ then x a single positive root (ψ_{12} suppose), derived from the quadratic equation of x,

$$x^2 - x(1 - u_1) - u_1 = 0$$

then $x = \frac{(1-u_1)+\sqrt{1+2u_1+u_1^2}}{2}$,

$T_2(0, c) = 0$ then c a single positive root (ψ_{13} suppose), derived from the cubic equation of c,

$$F_{11}c^3 + F_{12}c^2 + F_{13}c - F_{14} = 0$$

where
$F_{11} = v_1u_1v_3\beta_3/u_8$,
$F_{12} = v_1u_1(v_1u_1u_5 - \beta_2) - (v_1u_6 + v_3)u_1\beta_3/u_8$,
$F_{13} = \alpha_1 + u_1\beta_2 - v_1u_1u_5(2u_1 + u_4)$,
$F_{14} = q_0 - u_1u_5(u_1 + u_5)$.
Here, $F_{11} > 0$, $F_{12} > 0$, $F_{13} > 0$ and $F_{14} > 0$.
$T_2(x, 0) = 0$ then x a single positive root (ψ_{14} suppose), derived from the fourth degree equation of x,

$$F_{15}x^4 + F_{16}x^3 + F_{17}x^2 - F_{18}x + F_{19} = 0$$

where
$F_{15} = u_4$,
$F_{16} = u_2 + (u_1 - 1)(1 + u_4) - u_6/u_8$,
$F_{17} = (u_1 - 1) + u_4(u_5 - 1) + u_1u_4(4 - u_1)$,
$F_{18} = u_4(1 - u_1)(2u_1 + u_5) - u_5(u_1 + u_2)$.
$F_{19} = u_1u_5(u_1 + u_4)$.
Here, $F_{15} > 0$, $F_{16} > 0$, $F_{17} > 0$, $F_{18} > 0$ and $F_{19} > 0$.
Consequently, two isoclines cross each other in the area

$$M_2 = \{(x, y) : 0 < x < \psi_{14}, 0 < y < \psi_{11}\}$$

in the given two cases: (see Fig.5.1)

$$Case(i) : \psi_{11} > \psi_{13}, \psi_{14} > \psi_{12} \quad (62)$$

$$Case(ii) : \psi_{11} < \psi_{13}, \psi_{14} < \psi_{12} \quad (63)$$

This junction will result in $\bar{x}$, $\bar{y}$. For distinctiveness of $(\bar{x}, \bar{y})$, must have $\frac{dy}{dx} < 0$ for both of the region's curves of M_2.
For curve (29),

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$$\frac{dy}{dx} = \frac{\Phi_{32} - (v_1 c + z) + \frac{\Phi_{31}}{v_1}(1 - \frac{y}{(u_1+x)^2})}{\Phi_{34} + 1 - (u_6 - u_7 z - v_3 c) - \Phi_{33} - \frac{\Phi_{31}}{v_1(u_1+x)}} < 0 \tag{64}$$

and for curve (30),

$$\frac{dy}{dx} = \frac{\frac{\Phi_{35}}{v_1}(1 - \frac{y}{(u_1+x)^2}) - \beta_1 c}{\beta_2 c - (u_6 - u_7 z - v_3 c) - \frac{\Phi_{35}}{v_1(u_1+x)}} < 0 \tag{65}$$

where,
$\Phi_{31} = v_1(u_1 + x) + (v_2 x + v_3 y) - (x + u_7 y + u_8)\frac{q_0}{c^2\beta_3}$,
$\Phi_{32} = (y + u_5)(u_2 - u_3 y - v_2 c) + (1 - u_1 - 2x)$,
$\Phi_{33} = x(u_2 - u_3 y) - (u_4 y + v_2 cx)$,
$\Phi_{34} = (y + u_5)(u_4 + xu_3)$,
$\Phi_{35} = (\alpha_1 + x\beta_1 + y(\beta_2 + v_3) + z\beta_3) - (c\beta_3 + u_7 y + u_8)\frac{q_0}{c^2\beta_3}$.
In case (i), the exact value of $\frac{dy}{dx}$ given by (33) has a lower absolute value than $\frac{dy}{dx}$ given by (34). For the case (ii), The situation is the exact opposite.
Having knowledge of $\bar{x}$, $\bar{y}$; $\bar{z}$ and $\bar{c}$ is calculable from the equations (25)-(26).

4.3. Stability of Equilibria of Model 2

The eigenvalues of the Jacobian matrix surrounding each equilibrium are calculated in order to ascertain the local stability of each equilibrium in $Model$ 2.

According to $Model$ 2, the general variational matrix is

$$J_2 = \begin{bmatrix} o_{11} & -\frac{x}{x+u_1} & 0 & -xv_1 \\ \frac{u_4 y^2}{x^2} & o_{22} & -\frac{y}{y+u_5} & -yv_2 \\ 0 & \frac{u_8 z^2}{y^2} & -z(\frac{u_8}{y}) + j_{33} & -zv_3 \\ -\beta_1 c & -\beta_2 c & -\beta_3 c & -(\alpha_1 + \beta_1 x + \beta_2 y + \beta_3 z) \end{bmatrix}$$

where, $o_{11} = -x(1 - \frac{y}{(u_1+x)^2}) + j_{11}$, $o_{22} = -y(\frac{u_4}{x} - \frac{z}{(u_5+y)^2}) + j_{22}$, $j_{11} = 1 - x - \frac{y}{u_1+x} - v_1 c$, $j_{22} = u_2 - \frac{u_4 y}{x} - \frac{z}{u_5+y} - v_2 c$, $j_{33} = u_6 - \frac{u_8 z}{y} - v_3 c$.
In light of this, the examination of linear stability regarding the equilibrium points $E_{20}, \ddot{\ddot{E}}_{21}, \tilde{\tilde{E}}_{22}$ and $\bar{\bar{E}}_{23}$ produces the following outcomes:

- **For the equilibrium point** $\ddot{\ddot{E}}_{21}$, two eigenvalues of the characteristic equation are $u_2 - v_2\ddot{\ddot{c}}$ and $u_6 - v_3\ddot{\ddot{c}}$ and the other two eigenvalues are given by the roots of the following equation:

$$\lambda^2 + \lambda P_1 + P_2 = 0 \tag{66}$$

where,
$P_1 = \alpha_1 + (1 - v_1\ddot{\ddot{x}})(1 + \beta_1) > 0$ and $P_2 = (1 - v_1\ddot{\ddot{x}})(\alpha_1 + \beta_1 - 2\beta_1 v_1\ddot{\ddot{c}}) > 0$.
Equation (66)'s criterion is used to make the observed and providing that P_1 and P_2 are positive, $\ddot{\ddot{E}}_{21}$ is locally stable, *i.e.*, $(\alpha_1 + \beta_1\ddot{\ddot{x}}) > \beta_1 v_1\ddot{\ddot{c}}$ and $\ddot{\ddot{c}} > \frac{u_2}{v_2}$, $\ddot{\ddot{c}} > \frac{u_6}{v_3}$ and $2\alpha < \beta K_1$.
Remark 2: Same as given in *Remark 1*.

- **At the point** $\tilde{\tilde{E}}_{22}$, the characteristic equation's one eigenvalue is $u_6 - v_3\tilde{\tilde{c}}$ *i.e.*,

$$\tilde{\tilde{c}} > \frac{u_6}{v_3}$$

and the roots of the following equation provide the values for the remaining three eigenvalues.

$$\lambda^3 + \lambda^2 S_1 + \lambda S_2 + S_3 = 0 \tag{67}$$

where,
$S_1 = \alpha_1 + \beta_1\tilde{\tilde{x}} + \beta_2\tilde{\tilde{y}} + \tilde{\tilde{x}}(1 - \frac{\tilde{\tilde{y}}}{(u_1+\tilde{\tilde{x}})^2}) + \frac{u_4\tilde{\tilde{y}}}{\tilde{\tilde{x}}} > 0$,

$S_2 = \frac{u_4\tilde{\tilde{y}}^2}{\tilde{\tilde{x}}(u_1+\tilde{\tilde{x}})} + u_4\tilde{\tilde{y}}(1-\frac{\tilde{\tilde{y}}}{(u_1+\tilde{\tilde{x}})^2}) + (\alpha_1+\beta_1\tilde{\tilde{x}}+\beta_2\tilde{\tilde{y}})(\tilde{\tilde{x}}(1-\frac{\tilde{\tilde{y}}}{(u_1+\tilde{\tilde{x}})^2}) + \frac{u_4\tilde{\tilde{y}}}{\tilde{\tilde{x}}}) - \tilde{\tilde{x}}\tilde{\tilde{c}}v_1\beta_1 - \tilde{\tilde{y}}\tilde{\tilde{c}}v_2\beta_2 > 0,$

$S_3 = (\frac{u_4\tilde{\tilde{y}}^2}{\tilde{\tilde{x}}(u_1+\tilde{\tilde{x}})} + u_4\tilde{\tilde{y}}(1-\frac{\tilde{\tilde{y}}}{(u_1+\tilde{\tilde{x}})^2}))(\alpha_1+\beta_1\tilde{\tilde{x}}+\beta_2\tilde{\tilde{y}}) + \tilde{\tilde{y}}\tilde{\tilde{c}}v_2\beta_1\frac{\tilde{\tilde{x}}}{u_1+\tilde{\tilde{x}}} - \tilde{\tilde{y}}\tilde{\tilde{c}}v_1\beta_1u_4 - \tilde{\tilde{c}}v_1\beta_2\frac{u_4\tilde{\tilde{y}}^2}{\tilde{\tilde{x}}} - \tilde{\tilde{x}}\tilde{\tilde{y}}\tilde{\tilde{c}}v_2\beta_2(1-\frac{\tilde{\tilde{y}}}{(u_1+\tilde{\tilde{x}})^2}) > 0.$

Equation using the Routh-Hurwitz criterion (67), it has been noted that $\tilde{\tilde{E}}_{22}$ is locally asymptotically stable provided that S_1, S_2 and S_3 are positive and $S_1S_2 > S_3$ that is,
$(\alpha_1+\beta_1\tilde{\tilde{x}}+\beta_2\tilde{\tilde{y}})^2(\tilde{\tilde{x}}(1-\frac{\tilde{\tilde{y}}}{(u_1+\tilde{\tilde{x}})^2}) + \frac{u_4\tilde{\tilde{y}}}{\tilde{\tilde{x}}}) - \tilde{\tilde{x}}\tilde{\tilde{c}}v_1\beta_1(\alpha_1+\beta_1\tilde{\tilde{x}}+\beta_2\tilde{\tilde{y}})\ (\tilde{\tilde{x}}(1-\frac{\tilde{\tilde{y}}}{(u_1+\tilde{\tilde{x}})^2}) + \frac{u_4\tilde{\tilde{y}}}{\tilde{\tilde{x}}})(\frac{u_4\tilde{\tilde{y}}^2}{\tilde{\tilde{x}}(u_1+\tilde{\tilde{x}})} + u_4\tilde{\tilde{y}}(1-\frac{\tilde{\tilde{y}}}{(u_1+\tilde{\tilde{x}})^2}) + (\alpha_1+\beta_1\tilde{\tilde{x}}+\beta_2\tilde{\tilde{y}})(\tilde{\tilde{x}}(1-\frac{\tilde{\tilde{y}}}{(u_1+\tilde{\tilde{x}})^2}) + \frac{u_4\tilde{\tilde{y}}}{\tilde{\tilde{x}}}) - \tilde{\tilde{x}}\tilde{\tilde{c}}v_1\beta_1 - \tilde{\tilde{y}}\tilde{\tilde{c}}v_2\beta_2) + \tilde{\tilde{c}}v_1\beta_2\frac{u_4\tilde{\tilde{y}}^2}{\tilde{\tilde{x}}} + \tilde{\tilde{y}}\tilde{\tilde{c}}v_1\beta_1u_4 - \tilde{\tilde{y}}\tilde{\tilde{c}}v_2\beta_1\frac{\tilde{\tilde{x}}}{u_1+\tilde{\tilde{x}}} > 0.$

- For the equilibrium point's defining equation $\bar{\bar{E}}_{23}$, is shown as:

$$\lambda^4 + \lambda^3N_1 + \lambda^2N_2 + \lambda N_3 + N_4 = 0 \tag{68}$$

where,
$N_1 = n_1 + n_4 + n_7 + n_8$

$N_2 = n_1n_8 + (n_1+n_8)(n_4+n_7) + n_2n_3 + n_5n_6 + n_4n_7 - \bar{\bar{x}}\bar{\bar{c}}v_1\beta_1 - \bar{\bar{y}}\bar{\bar{c}}v_2\beta_2 - \bar{\bar{z}}\bar{\bar{c}}v_3\beta_3$

$N_3 = (n_1+n_8)(n_5n_6+n_4n_7) + (n_4+n_7)(n_1n_8-\bar{\bar{x}}\bar{\bar{c}}v_1\beta_1) - (n_1+n_7)\bar{\bar{y}}\bar{\bar{c}}v_2\beta_2 - (n_1+n_4)\bar{\bar{z}}\bar{\bar{c}}v_3\beta_3 + \bar{\bar{y}}\bar{\bar{c}}v_2n_2\beta_1 - \bar{\bar{x}}\bar{\bar{c}}v_1n_3\beta_2 + \bar{\bar{z}}\bar{\bar{c}}v_3n_5\beta_2 - \bar{\bar{y}}\bar{\bar{c}}v_2n_6\beta_3$

$N_4 = (n_1n_4+n_2n_3)(n_7n_8-\bar{\bar{z}}\bar{\bar{c}}v_3\beta_3) - \bar{c}(n_7\beta_2+n_6\beta_3)(\bar{\bar{x}}v_1n_3+\bar{\bar{y}}n_1v_2) + n_1n_5(\bar{\bar{z}}\bar{\bar{c}}v_3\beta_2+n_6n_8) + \bar{\bar{c}}n_2\beta_1(\bar{\bar{y}}v_2n_7-n_5n_7) - \bar{\bar{x}}\bar{\bar{c}}v_1\beta_1(n_5n_6+n_4n_7)$
and
$n_1 = \bar{\bar{x}}(1-\frac{\bar{\bar{y}}}{(u_1+\bar{\bar{x}})^2}), n_2 = \frac{\bar{\bar{x}}}{\bar{\bar{x}}+u_1}, n_3 = \frac{u_4\bar{\bar{y}}^2}{\bar{\bar{x}}^2}, n_4 = y(\frac{u_4}{\bar{\bar{x}}} - \frac{\bar{\bar{z}}}{(u_5+\bar{\bar{y}})^2}), n_5 = \frac{\bar{\bar{y}}}{\bar{\bar{y}}+u_5}, n_6 = \frac{u_8\bar{\bar{z}}^2}{\bar{\bar{y}}^2}, n_7 = \frac{\bar{\bar{z}}u_8}{\bar{\bar{y}}}, n_8 = \alpha_1 + \beta_1\bar{\bar{x}} + \beta_2\bar{\bar{y}} + \beta_3\bar{\bar{z}}.$
Using Routh - Hurwitz condition, $\bar{\bar{E}}_{23}$ is locally asymptotically stable provided that N_1, N_2, N_3 and N_4 and $N_1N_2N_3 > N_3^2 + N_1^2N_4$ are positive.

4.3.1. Global Stability

The point $\bar{\bar{E}}_{23}$'s global stability will be demonstrated in this section. With the aid of the proper Lyapunov function, we demonstrate the consequence of global stability. The global stability of the point $\bar{\bar{E}}_{23}$ guarantees that all the trajectories, starting from any point within the positive octant, finally approach the equilibrium point.
Theorem 5.1:
If any of the following disparities exist in the area Ω_2,

$$\bar{y} < \sigma_{21} \tag{69}$$

$$z\sigma_{22} < u_4\bar{x}\sigma_{23} \tag{70}$$

$$6[\frac{u_1+\bar{x}}{\sigma_{21}} - \frac{u_4\bar{y}}{\sigma_{22}}]^2 < (1-\frac{\bar{y}}{\sigma_{21}})(\frac{u_4\bar{x}}{\sigma_{22}} - \frac{W_l}{\sigma_{23}}) \tag{71}$$

$$6[v_1+\beta_1W_l]^2 < (1-\frac{\bar{y}}{\sigma_{21}})(\alpha_1+\beta_1\bar{x}+\beta_2\bar{y}+\beta_3\bar{z}) \tag{72}$$

$$6[\frac{1}{\sigma_{23}}(\bar{y}+u_5) - \frac{u_8}{\sigma_{24}}]^2 < \frac{u_8\bar{y}}{\sigma_{24}}(\frac{u_4\bar{x}}{\sigma_{22}} - \frac{W_l}{\sigma_{23}}) \tag{73}$$

$$9[v_2+\beta_2W_l]^2 < (\frac{u_4\bar{x}}{\sigma_{22}} - \frac{W_l}{\sigma_{23}})(\alpha_1+\beta_1\bar{x}+\beta_2\bar{y}+\beta_3\bar{z}) \tag{74}$$

$$6\sigma_{24}(v_3+\beta_3W_l)^2 < u_8\bar{y}(\alpha_1+\beta_1\bar{x}+\beta_2\bar{y}+\beta_3\bar{z}) \tag{75}$$

where, $\sigma_{21} = (u_1+\bar{x})(u_1+1)$, $\sigma_{22} = \bar{x}$, $\sigma_{23} = (u_5+\bar{y})(u_5+y_m)$ and $\sigma_{24} = \bar{y}W_l$, therefore the solutions starting in Ω_2 have a global asymptotic stability with respect to $\bar{\bar{E}}_{23}$.

Proof: We take into account the following non negative definite function regarding $\bar{\bar{E}}_{23}$:

$$V_{21} = (x - \bar{x} - \bar{x}ln(\frac{x}{\bar{x}})) + (y - \bar{y} - \bar{y}ln(\frac{y}{\bar{y}})) + (z - \bar{z} - \bar{z}ln(\frac{z}{\bar{z}})) + \frac{1}{2}(c - \bar{c})^2$$

When we differentiating V_{21} in relation to time t, we get

$$\frac{dV_{21}}{dt} = (\frac{x-\bar{x}}{x})\frac{dx}{dt} + (\frac{y-\bar{y}}{y})\frac{dy}{dt} + (\frac{z-\bar{z}}{z})\frac{dz}{dt} + (c-\bar{c})\frac{dc}{dt}$$

Equations (6)-(9), we pursue certain algebraic operations

$$\begin{aligned}\frac{dV_{21}}{dt} &= -(x-\bar{x})^2(1-\frac{\bar{y}}{\sigma_{21}}) - (y-\bar{y})^2(\frac{u_4\bar{x}}{\sigma_{22}} - \frac{z}{\sigma_{23}}) \\ &\quad -(z-z)^2(\frac{u_8\bar{x}}{\sigma_{24}}) - (c-\bar{c})^2(\alpha_1 + \beta_1\bar{x} + \beta_2\bar{y} + \beta_3\bar{z}) \\ &\quad -(x-\bar{x})(y-\bar{y})(\frac{u_1+\bar{x}}{\sigma_{21}} - \frac{u_4\bar{y}}{\sigma_{22}}) - (x-\bar{x})(c-\bar{c})(v_1+\beta_1 c) \\ &\quad -(y-\bar{y})(z-\bar{z})(\frac{1}{\sigma_{23}}(\bar{y}+u_5) - \frac{u_8}{\sigma_{24}}) \\ &\quad -(y-\bar{y})(c-\bar{c})(v_2+\beta_2 c) - (z-\bar{z})(c-\bar{c})(v_3+\beta_3 c)\end{aligned}$$

Where, $\sigma_{21} = (u_1+\bar{x})(u_1+x)$, $\sigma_{22} = x\bar{x}$, $\sigma_{23} = (u_5+\bar{y})(u_5+y)$ and $\sigma_{24} = y\bar{y}$.
Now $\frac{dV_{21}}{dt}$ additionally be expressed as the sum of the quadratic forms as

$$\begin{aligned}\frac{dV_{21}}{dt} &\leq -[(\frac{e_{11}}{2}(x-\bar{x})^2 + e_{12}(x-\bar{x})(y-\bar{y}) + \frac{e_{22}}{2}(y-\bar{y})^2) \\ &\quad +(\frac{e_{11}}{2}(x-\bar{x})^2 + e_{14}(x-\bar{x})(c-\bar{c}) + \frac{e_{44}}{2}(c-\bar{c})^2) \\ &\quad +(\frac{e_{22}}{2}(y-\bar{y})^2 + e_{23}(y-\bar{y})(z-\bar{z}) + \frac{e_{33}}{2}(z-\bar{z})^2) \\ &\quad +(\frac{e_{22}}{2}(y-\bar{y})^2 + e_{24}(y-\bar{y})(c-\bar{c}) + \frac{e_{44}}{2}(c-\bar{c})^2) \\ &\quad +(\frac{e_{33}}{2}(z-\bar{z})^2 + e_{34}(z-\bar{z})(c-\bar{c}) + \frac{e_{44}}{2}(c-\bar{c})^2)]\end{aligned}$$

where,
$e_{11} = (1-\frac{\bar{y}}{\sigma_{21}})/2$, $e_{12} = \frac{u_1+\bar{x}}{\sigma_{21}} - \frac{u_4\bar{y}}{\sigma_{22}}$, $e_{14} = v_1+\beta_1 c$, $e_{22} = (\frac{u_4\bar{x}}{\sigma_{22}} - \frac{z}{\sigma_{23}})/3$, $e_{23} = \frac{1}{\sigma_{23}}(\bar{y}+u_5) - \frac{u_8}{\sigma_{24}}$, $e_{24} = v_2+\beta_2 c$, $e_{33} = (\frac{u_8\bar{y}}{\sigma_{24}})/2$, $e_{34} = v_3+\beta_3 c$, $e_{44} = (\alpha_1+\beta_1\bar{x}+\beta_2\bar{y}+\beta_3\bar{z})/3$.
Now, by using Sylvesters criteria, we get $\frac{dV_{21}}{dt}$ is negative definite under the following conditions:

$$e_{11} > 0 \quad (76)$$
$$e_{22} > 0 \quad (77)$$
$$e_{11}e_{22} > e_{12}^2 \quad (78)$$
$$e_{11}e_{44} > e_{14}^2 \quad (79)$$
$$e_{22}e_{33} > e_{23}^2 \quad (80)$$
$$e_{22}e_{44} > e_{24}^2 \quad (81)$$
$$e_{33}e_{44} > e_{34}^2 \quad (82)$$

We note that the inequalities, (69) ⇒ (76), (70) ⇒ (77), (71) ⇒ (78), (72) ⇒ (79), (73) ⇒ (80), (74) ⇒ (81) and (75) ⇒ (82).
Hence V_{21} is a Lyapunov function with respect to $\bar{\bar{E}}_{23}$, whose realm includes the attraction's region Ω_2, the theorem being proved.

5. Analysis of Submodels of Model 2

5.1. Equilibria of Submodel 1 of Model 2

As considered the Submodel 1 of the main *Model* 2the following three neutral equilibrium states in x, y and c space specifically, $E_{31}^*(0,0,c^*)$, $\breve{E}_{32}(\breve{x},0,\breve{c})$, and $\breve{E}_{33}(\breve{x},\breve{y},\breve{c})$. We establish the validity of E_{31}^*, $\breve{E}_{32}$ and $\breve{E}_{33}$ as follows:

- **The existence at** $E^*_{31} = (0, 0, c^*)$
  from (12),

$$c^* = \frac{q_0}{\alpha_1} \tag{83}$$

- **The existence at** $\breve{E_{32}}(\breve{x}, 0, \breve{c})$,
  from (10),

$$\breve{x} = 1 - v_1\breve{c} \tag{84}$$

from (12) and (84),

$$\breve{c} = \frac{(\alpha_1 + \beta_1) \pm \sqrt{(\alpha_1 + \beta_1)^2 - 4q_0v_1\beta_1}}{2v_1\beta_1} \tag{85}$$

- **The existence at** $\breve{E_{33}}(\breve{x}, \breve{y}, \breve{c})$,
  from (10) and (11),

$$\breve{y} = \frac{\breve{x}(u_1 + \breve{x})(v_2(1 - \breve{x}) - v_1u_2)}{\breve{x}v_2 - v_1u_4(u_1 + \breve{x})} = h_1(x) \tag{86}$$

from (11),

$$\breve{c} = \frac{1}{v_2}(u_2 - \frac{u_4}{\breve{x}}h_1(x)) = h_2(x) \tag{87}$$

from (12), the polynomial of gets a positive root that we can use. x,

$$A_1\breve{x}^6 + A_2\breve{x}^5 + A_3\breve{x}^4 + A_4\breve{x}^3 + A_5\breve{x}^2 + A_6\breve{x} - A_7 = 0 \tag{88}$$

where the values of A_1 to A_7 involve equilibrium and model parameters.

5.2. Stability of Equilibria of Submodel 1 of Model 2

By calculating the eigenvalues of the variational matrix about each equilibrium, the local stability of equilibrium of submodel 1 of $Model$ 2 is ascertained.

Corresponds to the general variational matrix of submodel 1 of $Model$ 2 is

$$J_3 = \begin{bmatrix} k_{11} & -\frac{x}{x+u_1} & -xv_1 \\ \frac{u_4y^2}{x^2} & \frac{-u_4y}{x} + k_{22} & -yv_2 \\ -\beta_1c & -\beta_2c & -(\alpha_1 + \beta_1x + \beta_2y) \end{bmatrix}$$

where, $k_{11} = -x(1 - \frac{y}{(u_1+x)^2}) + m_{11}$, $k_{22} = u_2 - \frac{u_4y}{x} - v_2c$, $m_{11} = 1 - x - \frac{y}{u_1+x} - v_1c$.
In light of this, the examination of linear stability regarding $E^*_{31}, \breve{E_{32}}$ and $\breve{E_{33}}$ give the results below:

- **For the point** E^*_{31}, characteristic equation's eigenvalues are $1 - v_1c^*$, $u_2 - v_2c^*$ and $-\alpha_1$, which shows E^*_{31} is locally asymptotically stable if $\alpha_1 < q_0v_1$ and $\alpha_1u_2 < q_0v_2$ holds good.
- **For the point** $\breve{E_{32}}$, characteristic equation for $\breve{E_{32}}$ is given by

$$\lambda^3 + B_1\lambda^2 + B_2\lambda + B_3 = 0 \tag{89}$$

where,
$B_1 = (\alpha_1 + \beta_1\breve{x}) + \breve{x} - (u_2 - v_2\breve{c})$,
$B_2 = (\alpha_1 + \beta_1\breve{x})(\breve{x} - u_2) + \breve{c}v_2(\breve{x} + v_1) + \breve{x}\breve{c}\beta_1(v_2 - v_1) - \breve{x}u_2$,
$B_3 = \breve{x}(u_2 - v_2\breve{c})(\breve{c}v_1\beta_1 - (\alpha_1 + \beta_1\breve{x}))$.
According Routh-Hurwitz rule $\breve{E_{32}}$ is locally asymptotically stable if $B_1 > 0$, $B_2 > 0$, $B_3 > 0$ and $B_1B_2 > B_3$ that is,
$(\breve{x} - u_2)(\alpha_1 + \beta_1\breve{x} + \breve{c}v_2 - u_2) + \breve{x}\breve{c}\beta_1(v_2 - v_1) + \breve{c}v_2((\alpha_1 + \beta_1\breve{x}) - u_2) > 0$.

- **For the point** $\breve{E_{33}}$, The characteristic equation for $\breve{E_{33}}$ is given by

$$\lambda^3 + O_1\lambda^2 + O_2\lambda + O_3 = 0 \tag{90}$$

where,
$O_1 = x(1 - \frac{\breve{y}}{(u_1+\breve{x})^2}) + \frac{u_4\breve{y}}{\breve{x}} + (\alpha_1 + \beta_2(\breve{x} + \breve{y}))$,
$O_2 = \breve{y}u_4(1 - \frac{\breve{y}}{(u_1+\breve{x})^2}) + (\alpha_1 + \beta_2(\breve{x} + \breve{y}))(\breve{x}(1 - \frac{\breve{y}}{(u_1+\breve{x})^2}) + \frac{u_4\breve{y}}{\breve{x}}) + \frac{u_4\breve{y}^2}{\breve{x}(u_1+\breve{x})} - c\breve{y}v_2\beta_2 - \breve{c}\breve{x}v_1\beta_1$,
$O_3 = \breve{x}(1 - \frac{\breve{y}}{(u_1+\breve{x})^2})(\frac{u_4\breve{y}}{\breve{x}}(\alpha_1 + \beta_2(\breve{x} + \breve{y})) - \beta_2\breve{c}\breve{y}v_2) + \frac{u_4\breve{y}^2}{\breve{x}^2}(\frac{\breve{x}}{u_1+\breve{x}}(\alpha_1 + \beta_2(\breve{x} + \breve{y})) - \beta_2\breve{c}\breve{x}v_1) + \breve{c}\beta_1(\frac{\breve{x}\breve{y}v_2}{u_1+\breve{x}} - \breve{y}v_1u_4)$.
By the Routh-Hurwitz rule $\breve{E_{33}}$ is locally asymptotically stable if $O_1 > 0$, $O_2 > 0$, $O_3 > 0$ and $O_1O_2 > O_3$ that is,
$\breve{x}^2(1 - \frac{\breve{y}}{(u_1+\breve{x})^2})^2(\frac{u_4\breve{y}}{\breve{x}} + (\alpha_1 + \beta_2(\breve{x} + \breve{y}))) + \breve{x}(1 - \frac{\breve{y}}{(u_1+\breve{x})^2})(\frac{u_4\breve{y}^2}{\breve{x}(u_1+\breve{x})} - \breve{c}\breve{x}v_1\beta_1) - (\frac{u_4\breve{y}^2}{\breve{x}^2}(\frac{\breve{x}}{u_1+\breve{x}}(\alpha_1 + \beta_2(\breve{x} + \breve{y})) - \beta_2\breve{c}\breve{x}v_1) + \breve{c}\beta_1(\frac{\breve{x}\breve{y}v_2}{u_1+\breve{x}} - \breve{y}v_1u_4)) > 0$.

We may now attempt to determine the circumstances in which the system experiences Hopf-bifurcation. Due to its critical significance in the Holling type II functional response, which explains the predation of intermediate consumers, we chose the parameter u_4 as the bifurcation parameter for this reason. Now apply the Lius rule, [23] to determine the prerequisites for the Hopf-bifurcation-based small amplitude periodic solution.
As the equilibrium population densities are functions of u_4, the coefficients of the characteristic equation (90) are functions of the parameter u_4 and hence we can use the notation $O_i = O_i(u_4)$ for $i = 1, 2, 3$. Noting that the quantities O_is are smooth functions of the parameter u_4, we first state in our case, the definition of a simple Hopf-Bifurcation.
If a critical value u_4^* of parameter u_4 be found such that (i) a simple pair of complex conjugate eigenvalues of characteristic equation exists, say, $\lambda_1(u_4) = u(u_4) + iv(u_4)$, $\lambda_2(u_4) = u(u_4) - iv(u_4) = \overline{\lambda_1}(u_4)$. These eigen values will become purely imaginary at $u_4 = u_4^*$, *i.e.*, $\lambda_1(u_4^*) = iv_0$, $\lambda_2(u_4^*) = -iv_0$, with $v(u_4^*) = v_0 > 0$, and the other eigenvalue remains real and negative; and (ii) the transversality condition, $dRe\lambda_i(u_4^*)/du_4 \mid_{u_4=u_4^*} = du(u_4)/du_4 \mid_{u_4=u_4^*} \neq 0$ is satisfied. Then we find at $u_4 = u_4^*$, a simple Hopf-bifurcation. Without knowing eigenvalues, [23] proved that (referring the result to the current case): if $O_1(u_4), O_3(u_4), \Delta(u_4) = O_1(u_4)O_2(u_4) - O_3(u_4)$ are smooth functions of the parameter 'u_4' in an open interval containing $u_4^* \epsilon \Re^+$ such that following conditions hold:
(i_*) $O_1(u_4^*) > 0$, $\Delta(u_4^*) = 0$, $O_3(u_4^*) > 0$;
(ii_*) $d\Delta(u_4)/du_4 \mid_{u_4=u_4^*} \neq 0$
then (i_*) and (ii_*) are equivalent to conditions (i) and (ii) for the occurrence of a simple Hopf-bifurcation at $u_4 = u_4^*$. Hence we can propose the following theorem:
Theorem 3.1 If a critical value u_4^* of parameter u_4 be found such that $O_1(u_4^*) > 0$, $O_3(u_4^*) > 0$ and $\Delta(u_4^*) = 0$ and further $\Delta' \neq 0$ (where prime denotes differentiation with respect to u_4) then system (10)-(12) undergoes Hopf-bifurcation around E_{33}.

5.3. Equilibria of Submodel 2 of Model 2

Now let us take Submodel 2 of *Model* 2, it has three non-negative equilibria following in y, z and c space namely, $E_{41}^{**}(0, 0, c^{**})$, $\breve{\breve{E}}_{42}(\breve{\breve{y}}, 0, \breve{\breve{c}})$, and $\breve{\breve{E}}_{43}(\breve{\breve{y}}, \breve{\breve{z}}, \breve{\breve{c}})$. We prove the existence of E_{41}^{**}, $\breve{\breve{E}}_{42}$ and $\breve{\breve{E}}_{43}$ as follows:

- **The existence at** $E_{41}^{**} = (0, 0, c^{**})$
from (5),

$$c^{**} = \frac{q_0}{\alpha_1} \tag{91}$$

- **The existence at** $\breve{\breve{E}}_{42}(\breve{\breve{y}}, 0, \breve{\breve{c}})$,
From (13),

$$\breve{\breve{y}} = \frac{\bar{\bar{x}}}{u_4}(u_2 - v_2\breve{\breve{c}}) \tag{92}$$

$\check{\check{y}} > 0$ if $u_2 > v_2\check{\check{c}}$.
From (15) and (92) we get,

$$Q_1\check{\check{c}}^2 - Q_2\check{\check{c}} + q_0 = 0 \tag{93}$$

where, $Q_1 = \frac{\bar{\bar{x}}v_2\beta_2}{u_4}, Q_2 = \alpha_1 + \bar{\bar{x}}\beta_1 + \frac{\bar{\bar{x}}u_2\beta_2}{u_4}$.
We get a positive root,

$$\check{\check{c}} = \frac{Q_2 \pm \sqrt{Q_2^2 - 4Q_1q_0}}{2Q_1} \tag{94}$$

- **The existence at** $\check{\check{E}}_{43}(\check{\check{y}}, \check{\check{z}}, \check{\check{c}})$,
from (13) and (14),

$$\check{\check{z}} = \frac{v_3(u_2 - \frac{\check{\check{y}}u_4}{\bar{\bar{x}}}) - v_2u_6}{(\frac{v_3}{\check{\check{y}}+u_5} - \frac{v_2u_8}{\check{\check{y}}})} = w_1(y) \tag{95}$$

from (14),

$$\check{\check{c}} = \frac{1}{v_3(u_6 - \frac{u_8}{\check{\check{y}}}w_1(y))} = w_2(y) \tag{96}$$

using (95) and (96) in (15) we get a positive root from the polynomial of $\check{\check{y}}$,

$$L_1\check{\check{y}}^5 + L_2\check{\check{y}}^4 + L_3\check{\check{y}}^3 + L_4\check{\check{y}}^2 + L_5\check{\check{y}} - L_6 = 0 \tag{97}$$

where the values of L_1 to L_6 involve equilibrium and model parameters.

5.4. Stability of Equilibria of Submodel 2 of Model 2

Computing the eigenvalues of the variational matrix for each equilibrium allows one to ascertain the local stability of equilibrium of Submodel 2.

Corresponding to the general variational matrix the submodel 2 of $Model$ 2 is

$$J_4 = \begin{bmatrix} p_{11} & -\frac{y}{y+u_5} & -yv_2 \\ \frac{u_8z^2}{y^2} & u_6 - \frac{2u_8z}{y} - v_3c & -zv_3 \\ -\beta_2c & -\beta_3c & -(\alpha_1 + \beta_1\bar{\bar{x}} + \beta_2y + \beta_3z) \end{bmatrix}$$

where, $p_{11} = u_2 - \frac{2u_4y}{\bar{\bar{x}}} - \frac{zu_5}{(u_5+y)^2} - v_2c$.
The linear stability analysis for $E_{41}^{**}, \check{\check{E}}_{42}$ and $\check{\check{E}}_{43}$ are given as follows:

- The point E_{41}^{**}, the eigenvalues of the characteristic equation are $u_2 - v_2c^{**}$, $u_6 - v_3c^{**}$ and $-(\alpha_1+\bar{\bar{x}}\beta_1)$, which shows E_{41}^{**} is locally asymptotically stable if $\alpha_1u_2 < q_0v_2$ and $\alpha_1u_6 < q_0v_3$ holds good.

- The point $\check{\check{E}}_{42}$, one eigenvalue of the characteristic equation is $u_6 - v_3\check{\check{c}}$ and other two eigenvalues are given by the roots of the following equation

$$\lambda^2 + \lambda(\alpha_1 + \bar{\bar{x}}\beta_1 + \frac{\check{\check{y}}u_4}{\bar{\bar{x}}}) + \check{\check{y}}((\alpha_1 + \bar{\bar{x}}\beta_1)\frac{u_4}{\bar{\bar{x}}} - \check{\check{c}}v_2\beta_2) = 0 \tag{98}$$

By the Routh-Hurwitzs rule it is observed that $\check{\check{E}}_{42}$ is locally asymptotically stable if the conditions hold well.

$$\check{\check{c}} > \frac{u_6}{v_3} \tag{99}$$

and

$$\check{\check{c}}v_2\bar{\bar{x}}\beta_2 < u_4(\alpha_1 + \bar{\bar{x}}\beta_1). \tag{100}$$

6 NUMERICAL SIMULATIONS:

- The point $\breve{\breve{E}}_{43}$, The characteristic equation about $\breve{\breve{E}}_{43}$ is given by

$$\lambda^3 + I_1\lambda^2 + I_2\lambda + I_3 = 0 \tag{101}$$

where,
$I_1 = (\frac{u_4}{\bar{\bar{x}}} - \frac{\breve{\breve{z}}}{(\breve{y}+u_5)^2}) + (\frac{\breve{\breve{z}}u_8}{\breve{y}}) - (\alpha_1 + \beta_1\bar{\bar{x}} + \beta_2\breve{y} + \beta_3\breve{\breve{z}})$,
$I_2 = (\frac{\breve{\breve{z}}u_8}{\breve{y}})((\frac{u_4}{\bar{\bar{x}}} - \frac{\breve{\breve{z}}}{(\breve{y}+u_5)^2}) - (\alpha_1 + \beta_1\bar{\bar{x}} + \beta_2\breve{y} + \beta_3\breve{\breve{z}})) - (\frac{u_4}{\bar{\bar{x}}} - \frac{\breve{\breve{z}}}{(\breve{y}+u_5)^2})(\alpha_1 + \beta_1\bar{\bar{x}} + \beta_2\breve{y} + \beta_3\breve{\breve{z}}) + \breve{c}(\breve{\breve{z}}v_3\beta_3 + \breve{y}v_2\beta_2) - \frac{\breve{\breve{z}}^2 u_8}{\breve{y}(u_5+\breve{y})}$,
$I_3 = \breve{\breve{z}}\breve{c}v_2u_8(\frac{\breve{\breve{z}}\beta_3}{\breve{y}} + \beta_2) + \breve{c}\breve{\breve{z}}v_3\beta_3(\frac{u_4}{\bar{\bar{x}}} - \frac{\breve{\breve{z}}}{(\breve{y}+u_5)^2}) - (\frac{u_4}{\bar{\bar{x}}} - \frac{\breve{\breve{z}}}{(\breve{y}+u_5)^2})(\frac{\breve{\breve{z}}u_8}{\breve{y}})(\alpha_1 + \beta_1\bar{\bar{x}} + \beta_2\breve{y} + \beta_3\breve{\breve{z}}) - (\frac{\breve{y}}{u_5+\breve{y}})((\breve{c}\breve{\breve{z}}v_2\beta_2) + (\frac{\breve{\breve{z}}^2 u_8}{\breve{y}^2})(\alpha_1 + \beta_1\bar{\bar{x}} + \beta_2\breve{y} + \beta_3\breve{\breve{z}}))$.
By Routh–Hurwitz rule $\breve{\breve{E}}_{43}$ is locally asymptotically stable if $I_1 > 0$, $I_2 > 0$, $I_3 > 0$ and $I_1I_2 > I_3$ that is,
$(\frac{\breve{\breve{z}}u_8}{\breve{y}}) - (\alpha_1+\beta_1\bar{\bar{x}}+\beta_2\breve{y}+\beta_3\breve{\breve{z}})((\frac{\breve{\breve{z}}u_8}{\breve{y}})((\frac{u_4}{\bar{\bar{x}}} - \frac{\breve{\breve{z}}}{(\breve{y}+u_5)^2}) - (\alpha_1+\beta_1\bar{\bar{x}}+\beta_2\breve{y}+\beta_3\breve{\breve{z}})) - (\frac{u_4}{\bar{\bar{x}}} - \frac{\breve{\breve{z}}}{(\breve{y}+u_5)^2})(\alpha_1 + \beta_1\bar{\bar{x}} + \beta_2\breve{y} + \beta_3\breve{\breve{z}}) + (\breve{c}\breve{\breve{z}}v_3\beta_3) + (\breve{y}\breve{c}v_2\beta_2) - \frac{\breve{\breve{z}}^2 u_8}{\breve{y}(u_5+\breve{y})}) + (\frac{u_4}{\bar{\bar{x}}} - \frac{\breve{\breve{z}}}{(\breve{y}+u_5)^2})^2((\frac{\breve{\breve{z}}u_8}{\breve{y}}) - (\alpha_1 + \beta_1\bar{\bar{x}} + \beta_2\breve{y} + \beta_3\breve{\breve{z}})) + (\frac{u_4}{\bar{\bar{x}}} - \frac{\breve{\breve{z}}}{(\breve{y}+u_5)^2})((\breve{y}\breve{c}v_2\beta_2) - \frac{\breve{\breve{z}}^2 u_8}{\breve{y}(u_5+\breve{y})}) - \breve{\breve{z}}\breve{c}v_2u_8(\frac{\breve{\breve{z}}\beta_3}{\breve{y}} + \beta_2) + (\frac{\breve{y}}{u_5+\breve{y}})((\breve{c}\breve{\breve{z}}v_2\beta_2) + (\frac{\breve{\breve{z}}^2 u_8}{\breve{y}^2})(\alpha_1 + \beta_1\bar{\bar{x}} + \beta_2\breve{y} + \beta_3\breve{\breve{z}})) > 0$.

Now, let us make an attempt to observe the conditions under which the system undergoes Hopf-bifurcation. For this purpose, we select the parameter u_4 as bifurcation parameter as it plays a important role in Holling type II functional response which tells the predation of intermediate consumer. Now let us apply the Lius rule, [23] to get the conditions for small amplitude periodic solution arising from Hopf-bifurcation.
As the equilibrium population densities are functions of u_4, the coefficients of the characteristic equation (101) are functions of the parameter u_4 and hence we can use the notation $I_i = I_i(u_4)$ for $i = 1, 2, 3$. Noting that the quantities I_is are smooth functions of the parameter u_4, we first state in our case, the definition of a simple Hopf-Bifurcation.
If a critical value u_4^* of parameter u_4 be found such that (i) a simple pair of complex conjugate eigenvalues of characteristic equation exists, say, $\lambda_1(u_4) = u(u_4) + iv(u_4)$, $\lambda_2(u_4) = u(u_4) - iv(u_4) = \overline{\lambda_1}(u_4)$. These eigen values will become purely imaginary at $u_4 = u_4^*$, *i.e.*, $\lambda_1(u_4^*) = iv_0$, $\lambda_2(u_4^*) = -iv_0$, with $v(u_4^*) = v_0 > 0$, and the other eigenvalue remains real and negative; and (ii) the transversality condition, $dRe\lambda_i(u_4^*)/du_4 \mid_{u_4=u_4^*} = du(u_4)/du_4 \mid_{u_4=u_4^*} \neq 0$ is satisfied. Then we find at $u_4 = u_4^*$, a simple Hopf-bifurcation. Without knowing eigenvalues, [23] proved that (referring the result to the current case): if $I_1(u_4), I_3(u_4), \Delta(u_4) = I_1(u_4)I_2(u_4) - I_3(u_4)$ are smooth functions of the parameter 'u_4' in an open interval containing $u_4^*\epsilon\Re^+$ such that following conditions hold:
(i_*) $I_1(u_4^*) > 0$, $\Delta(u_4^*) = 0$, $I_3(u_4^*) > 0$;
(ii_*) $d\Delta(u_4)/du_4 \mid_{u_4=u_4^*} \neq 0$
then (i_*) and (ii_*) are equivalent to conditions (i) and (ii) for the occurrence of a simple Hopf-bifurcation at $u_4 = u_4^*$. Hence we can propose the following theorem:
Theorem 3.1 If a critical value u_4^* of parameter u_4 be found such that $I_1(u_4^*) > 0$, $I_3(u_4^*) > 0$ and $\Delta(u_4^*) = 0$ and further $\Delta' \neq 0$ (where prime denotes differentiation with respect to u_4) then system (13)-(15) undergoes Hopf-bifurcation around E_{43}.

6. Numerical Simulations:

In order to make it easier to understand our mathematical results, we present in this part a numerical simulation that shows the dynamical behaviour of a toxicant's effect on a three-species food-chain system when the population is negatively impacted by the toxicant. The graphs have been plotted with the aid of MATLAB, and the figures show how all of the equilibrium points of the models behave in terms of stability for the specified sets of parameters.

6 NUMERICAL SIMULATIONS:

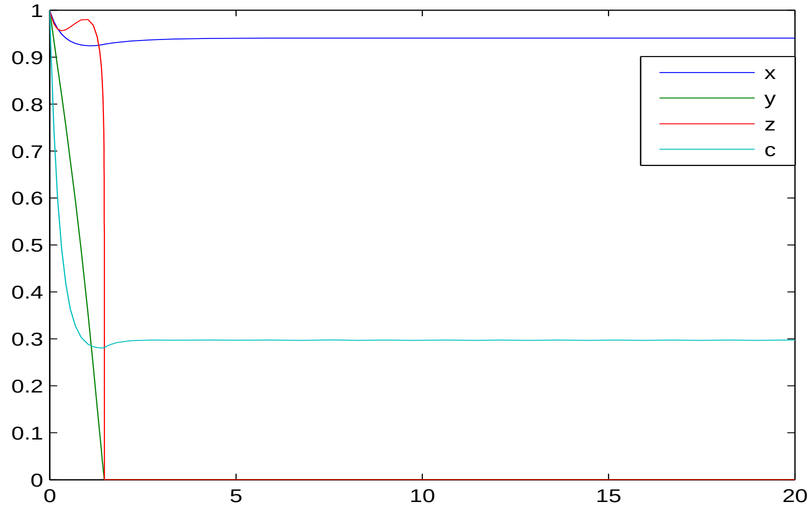


Figure 1. A time graph for Model 1 that shows stability around the point $\ddot{E}_{11}$.

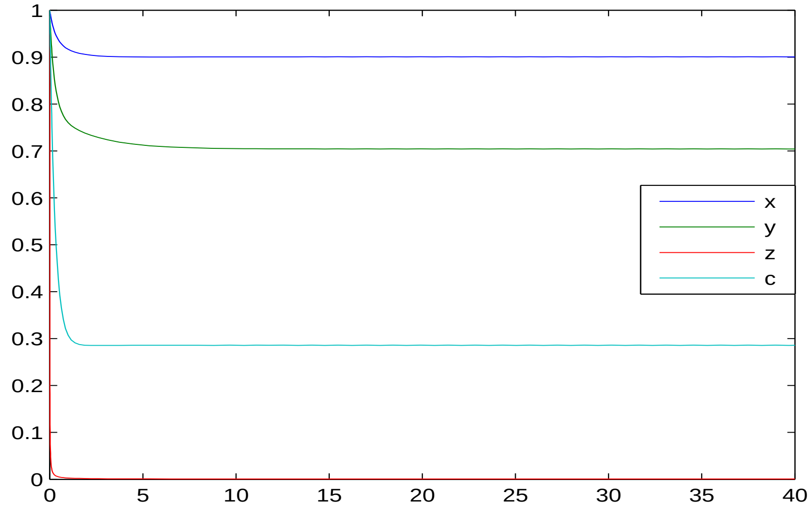


Figure 2. A time graph for Model 1 that shows stability around the point $\tilde{E}_{12}$.

6 NUMERICAL SIMULATIONS:

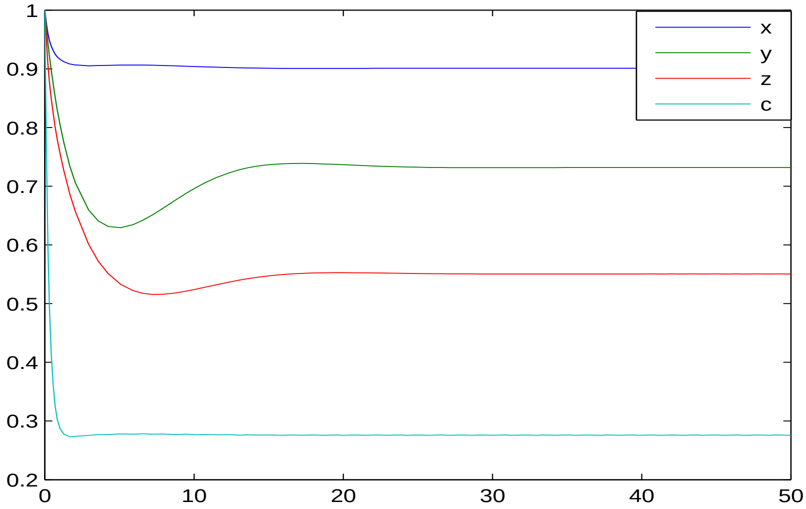


Figure 3. A time graph for Model 1 that shows stability around the point $\bar{E}_{13}$.

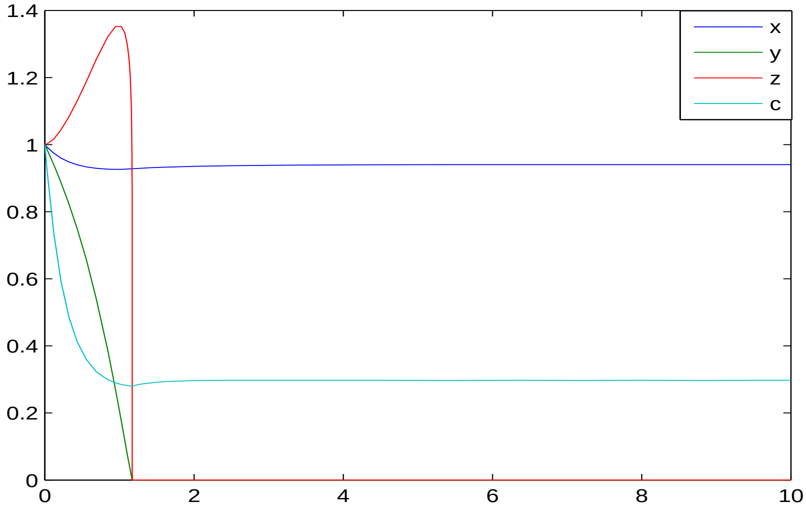


Figure 4. A time graph for Model 1 that shows stability around the point $\ddot{\ddot{E}}_{21}$.

6 NUMERICAL SIMULATIONS:

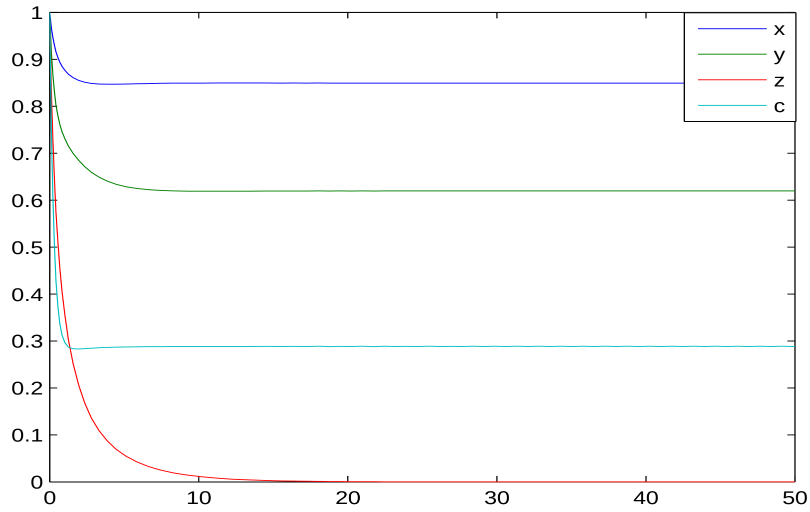


Figure 5. A time graph for Model 1 that shows stability around the point $\tilde{\tilde{E}}_{22}$.

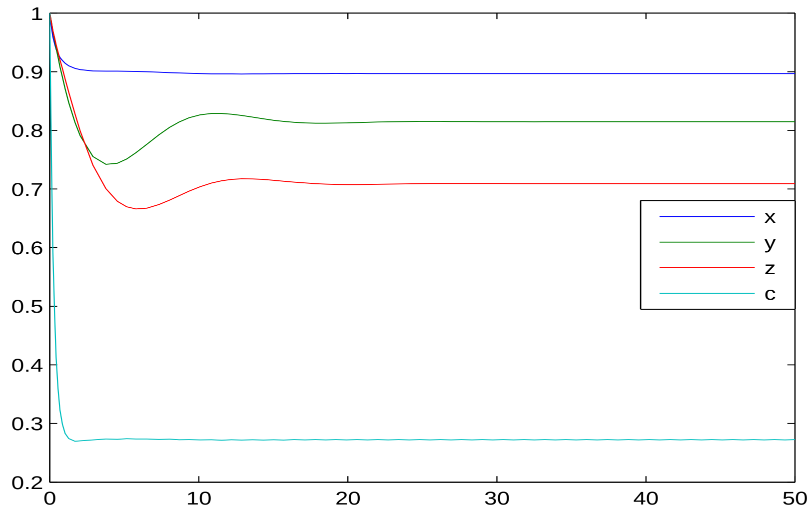


Figure 6. A time graph for Model 1 that shows stability around the point $\bar{\bar{E}}_{23}$.

6 NUMERICAL SIMULATIONS:

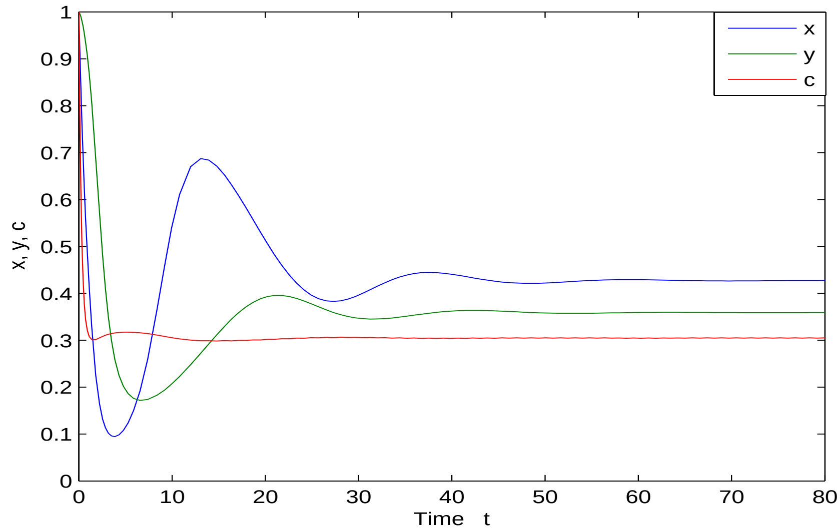


Figure 7. A time graph for Model 1 that shows stability around the point $\breve{E_{33}}$.

Table 1. Analysis experiment of Subsystem 1 of *Model* 2 for different values of parameter "q_0", *i.e.*, the external input of toxicant into environment.

Fig.	Parameter	Equilibrium Values of x y c
Fig.10	q_0=0.3899	0.4540, 0.3562, 0.1216
	q_0=0.5899	0.4561, 0.3556, 0.1842
	q_0=0.7899	0.4573, 0.3551, 0.2464

6.1. Numerical Simulation for Model 1

We choose the following values of parameters for $\ddot{E_{11}}$:

$$u_1 = 15.82; \quad u_2 = 0.23; \quad u_3 = 0.1; \quad u_4 = 0.21; \quad u_5 = 0.00001; \quad u_6 = 0.15;$$
$$u_7 = 0.4031; \quad u_8 = 0.104; \quad v_1 = 0.2001; \quad v_2 = 1; \quad v_3 = 1; \quad q_0 = 0.975;$$
$$\alpha_1 = 2.998; \quad \beta_1 = 0.301; \quad \beta_2 = 0.2012; \quad \beta_3 = 0.211.$$

It is observed that the above set of parameters, the point $\ddot{E_{11}}$

$$\ddot{x} = 0.9405, \quad \ddot{y} = 0.0000, \quad \ddot{z} = 0.0000, \quad \ddot{c} = 0.2973,$$

is locally asymptotically stable (see Fig.1).
We select the following set values for $\tilde{E_{12}}$:

$$u_3 = 0.401; \quad u_5 = 0.04; \quad u_6 = 0.14; \quad u_7 = 399.0; \quad u_8 = 5.969.$$

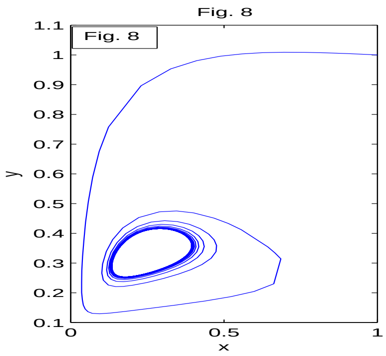
Along with the above values and selecting the remaining parameters to be the same as considered for $\ddot{E_{11}}$, it is observed that the above set of values for the point $\tilde{E_{12}}$

$$\tilde{x} = 0.9007, \quad \tilde{y} = 0.7043, \quad \tilde{z} = 0.0000, \quad \tilde{c} = 0.2858,$$

is locally asymptotically stable (see Fig.2).
We select the some values for $\bar{E_{13}}$:

$$u_5 = 0.401; \quad u_6 = 0.64; \quad u_7 = 0.41; \quad u_8 = 0.404; \quad v_3 = 0.40001.$$

6 NUMERICAL SIMULATIONS:



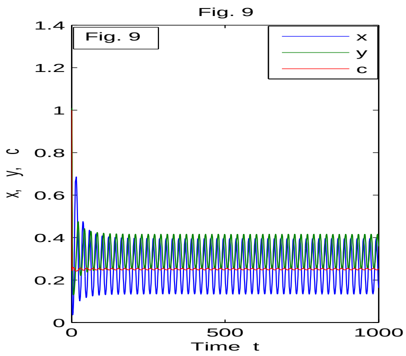


Figure 8. A time graph in the plane - XY for the Submodel 1 of Model 2 for the point $\breve{E}_{33}$, highlighting the bifurcation behavior.

Figure 9. A time graph for the Submodel 1 of Model 2 for the point $\breve{E}_{33}$, highlighting the bifurcation behavior

Table 2. Analysis experiment of Subsystem 2 of *Model* 2 for different values of parameter "q_0", *i.e.*, the external input of toxicant into environment.

Fig.	Parameter	Equilibrium Values of y, z, c
Fig.14	q_0=0.3899	2.3501, 2.0733, 0.1892
	q_0=0.5899	2.2561, 2.0054, 0.1423
	q_0=0.7899	2.1678, 1.9415, 0.0947

Along with the above values and taking the remaining parameters to be the same as taken for $\ddot{E}_{11}$, it is observed that the above set of values for the point $\bar{E}_{13}$

$$\ddot{x} = 0.9010, \quad \ddot{y} = 0.7319, \quad \ddot{z} = 0.5505, \quad \ddot{c} = 0.2758$$

is locally asymptotically stable (see Fig.3).

6.2. Numerical Simulation for Model 2

We select the these values for $\ddddot{E}_{21}$:

$$\begin{array}{lllll} u_1 = 15.82; & u_2 = 0.13; & u_3 = 0.1; & u_4 = 0.21; & u_5 = 0.00001; \\ u_7 = 0.4031; & u_8 = 0.104; & v_1 = 0.2001; & v_2 = 1; & v_3 = 1; \\ u_6 = 0.15; & q_0 = 0.975; & \alpha_1 = 2.998; & \beta_1 = 0.301; & \beta_2 = 0.2012; \beta_3 = 0.211. \end{array}$$

It has been found that the above set of parameters, the point $\ddddot{E}_{21}$

$$\ddddot{x} = 0.9405, \quad \ddddot{y} = 0.0000, \quad \ddddot{z} = 0.0000, \quad \ddddot{c} = 0.2972,$$

is locally asymptotically stable (see Fig.4).
We have taken the following values of parameters for $\tilde{\tilde{E}}_{22}$:

$$\begin{array}{llll} u_1 = 5.82; & u_4 = 0.61; & u_5 = 12.91; & u_6 = 0.00001; \\ u_8 = 0.804; & v_2 = 0.9872; & v_3 = 0.9858. & \end{array}$$

Along with the above values and taking the remaining parameters to be the same as taken for $\ddddot{E}_{21}$, it has been found that the above set of values, the point $\tilde{\tilde{E}}_{22}$

$$\tilde{\tilde{x}} = 0.8493, \quad \tilde{\tilde{y}} = 0.6197, \quad \tilde{\tilde{z}} = 0.0000, \quad \tilde{\tilde{c}} = 0.2884,$$

6 NUMERICAL SIMULATIONS:

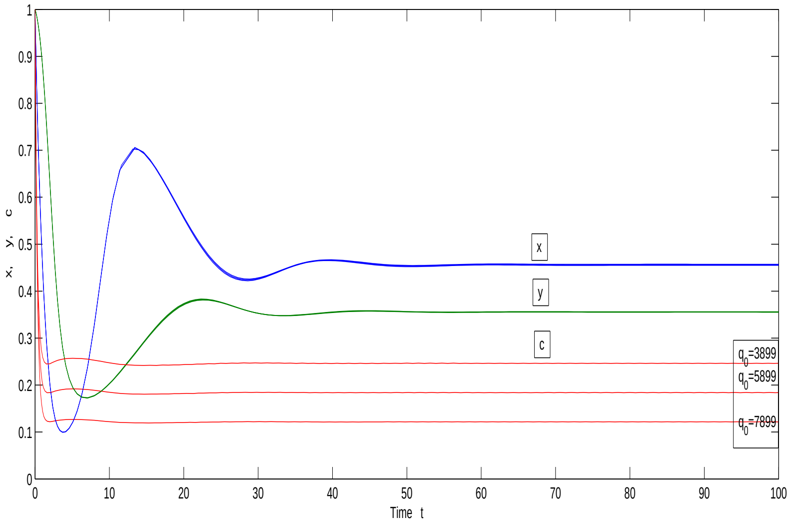


Figure 10. Contrast of x, y and c with respect to time t, corresponding to different values of q_0 in Submodel 1 of Model 2.

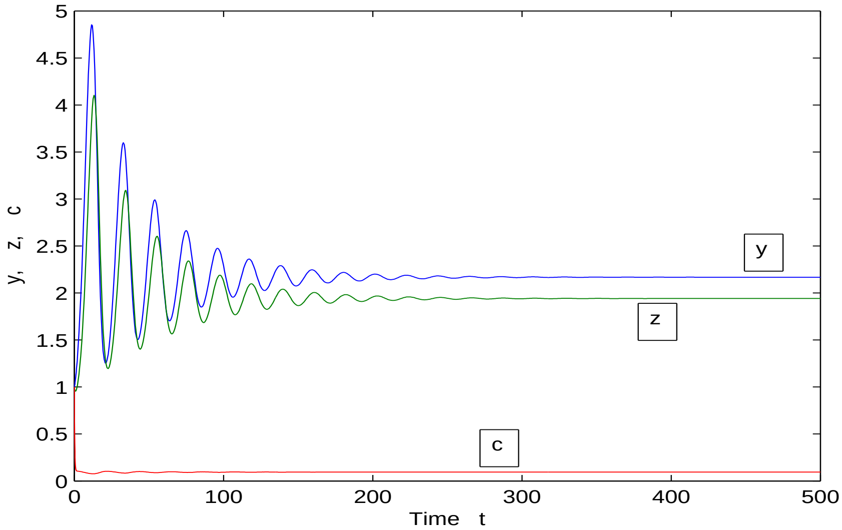


Figure 11. A time graph for the Submodel 2 of Model 2 for the point $\breve{\breve{E}}_{43}$, displaying the stability behavior.

6 NUMERICAL SIMULATIONS:

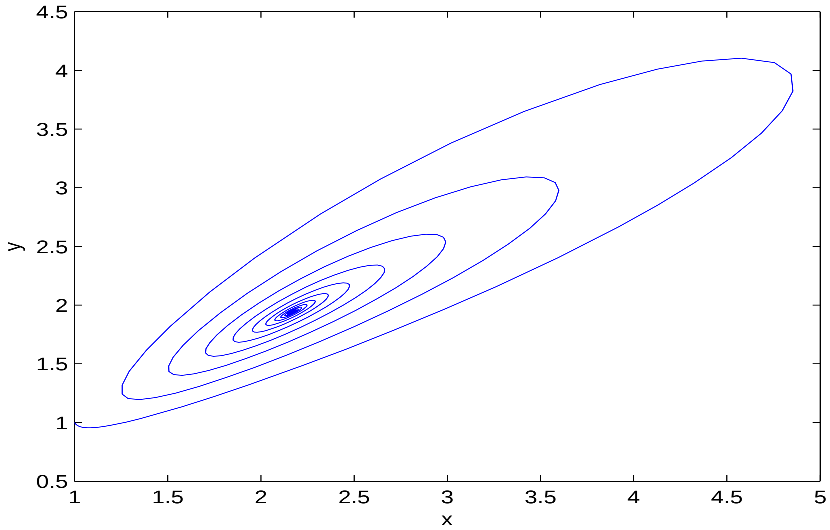


Figure 12. A time graph in the plane - XY for the Submodel 2 of Model 2 for the point $\breve{\breve{E}}_{43}$, displaying the stability behavior.

is locally asymptotically stable (see Fig.5).
We have chosen the following values of parameters for $\bar{\bar{E}}_{23}$:

$$u_5 = 0.501; \quad u_6 = 0.64; \quad u_8 = 0.704; \quad v_2 = 0.001; \quad v_3 = 0.1001.$$

Along with the these values and taking the rest of parameters to be the same as taken for $\ddot{\ddot{E}}_{21}$, it has been found that the above set of parameters, the point $\bar{\bar{E}}_{23}$

$$\bar{\bar{x}} = 0.8968, \quad \bar{\bar{y}} = 0.8147, \quad \bar{\bar{z}} = 0.7091, \quad \bar{\bar{c}} = 0.2725,$$

is locally asymptotically stable (see Fig.6).

6.3. Numerical Simulation for the Subsystem 1 of Model 2

Wef have chosen the these values of parameters for the first subsystem:

$$u_1 = 0.2; \quad u_2 = 0.16; \quad u_4 = 0.11; \quad v_1 = 0.001; \quad v_2 = 0.001; \quad q_0 = 0.975;$$
$$\alpha_1 = 2.998; \quad \beta_1 = 0.301; \quad \beta_2 = 0.2012.$$

It has been found that the above set of parameters, the point $\breve{E}_{33}(\breve{x}, \breve{y}, \breve{c})$

$$\breve{x} = 0.4275, \quad \breve{y} = 0.3590, \quad \breve{c} = 0.3052,$$

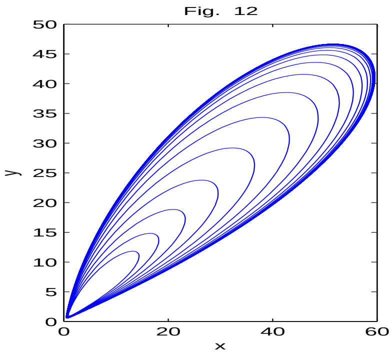
is locally asymptotically stable (see Fig.7).
Now, we observe the Hopf-bifurcation of the first subsystem, considering u_1 as the bifurcating parameter. The transversality rule holds with the set of parameters when $u_1 = \breve{u}_1 = 0.22$. It is very clear that the interior point $\breve{E}_{33}$ of first submodel is stable when $u_1 < \breve{u}_1$ and unstable when $u_1 > \breve{u}_1$ for which, the Hopf-bifurcation occurs (see Fig.8).

6.4. Numerical Simulation for the Subsystem 2 of Model 2

We select the following parameter values for the second subsystem:

$$u_2 = 0.73; \quad u_4 = 0.001; \quad u_5 = 0.501; \quad u_6 = 0.64; \quad u_8 = 0.704; \quad v_2 = 0.001;$$
$$v_3 = 0.1001; \quad q_0 = 0.7899; \quad \alpha_1 = 2.998; \quad \beta_1 = 0.301; \quad \beta_2 = 0.2012; \quad \beta_3 = 0.211.$$

7 CONCLUSION



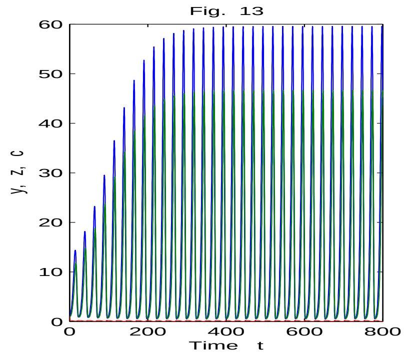


Figure 13. A time graph in the plane - XY for the Submodel 2 of Model 2 for the point $\breve{\breve{E}}_{43}$, displaying the bifurcation behavior.

Figure 14. A time graph for the Submodel 2 of Model 2 for the point $\breve{\breve{E}}_{43}$, displaying the bifurcation behavior.

It has been found that under the above set of parameters, the equilibrium point $\breve{\breve{E}}_{43}(\breve{\breve{y}}, \breve{\breve{z}}, \breve{\breve{c}})$

$$\breve{\breve{y}} = 0.4275, \quad \breve{\breve{z}} = 0.3590, \quad \breve{\breve{c}} = 0.3052,$$

is locally asymptotically stable (see Fig.9).
Now, we observe the Hopf-bifurcation of the Subsystem 2 of Model 2, taking u_2 as the bifurcating parameter. The transversality rule holds with the above set of parameters when $u_2 = \breve{\breve{u}}_2 = 0.75$. It is very clear that the interior equilibrium point $\breve{\breve{E}}_{43}$ of submodel 2 of Model 2 is stable when $u_2 < \breve{\breve{u}}_2$ and unstable when $u_2 > \breve{\breve{u}}_2$ for which Hopf-bifurcation occurs (see Fig.10).

7. Conclusion

In order to investigate the impact of pollution on a three species ratio dependent food chain system, a nonlinear mathematical model is proposed and examined in this research. The primary mathematical model is broken down into submodels and studies the effects of the pollution. It is assumed in the model that the pollutant has direct adverse effects on all biological species in the three species food chain system.

For the stability of $\ddot{E}_{11}$ and $\ddot{\ddot{E}}_{21}$, it is observed in the analysis that when toxicant concentrations are present, only prey populations will thrive and predator populations will generally become extinct. (see Fig.1 and 4). For $\tilde{E}_{12}$ and $\tilde{\tilde{E}}_{22}$, according to the findings, the top predator will likely go extinct in the presence of toxicant, leaving only the prey and intermediate predator groups to survive. However, the level of $\tilde{\tilde{E}}_{22}$ decreases due to the absence of intra-specific competition (crowding) (see Fig.2 and 5). For $\bar{E}_{13}$ of $Model$ 1 and $\bar{\bar{E}}_{23}$ of $Model$ 2 makes sure that even in the presence of toxicants, all three species can coexist. (see Fig.3 and 6).

Gakkhar and Naji [41] previously demonstrated the presence of chaotic dynamics in a three species to one food chain system, but in this study, chaotic behaviour and the limit cycle phenomenon are not seen for $Model$ 1 and $Model$ 2 when we consider the presence of pollutant in the system. Similar kind of results can be observed in Chattopadhyay and Sarkar [43], the authors have demonstrated that by boosting the level of pollution in the biological system, the prevalence of chaos, period doubling, and limit cycle oscillations may be decreased.

In Submodel 1 and 2 of $Model$ 2, the hopf bifurcation behaviour has been seen (see Figs.7-9,11-14). From the Table 1 and Fig.10, When the toxicant input rate is q_0 increases, the system can be seen to remain stable, that indicates that the toxicant causes the biological system to stabilise under the circumstances described in the Submodel 1 of Model 2's section on stability. From the

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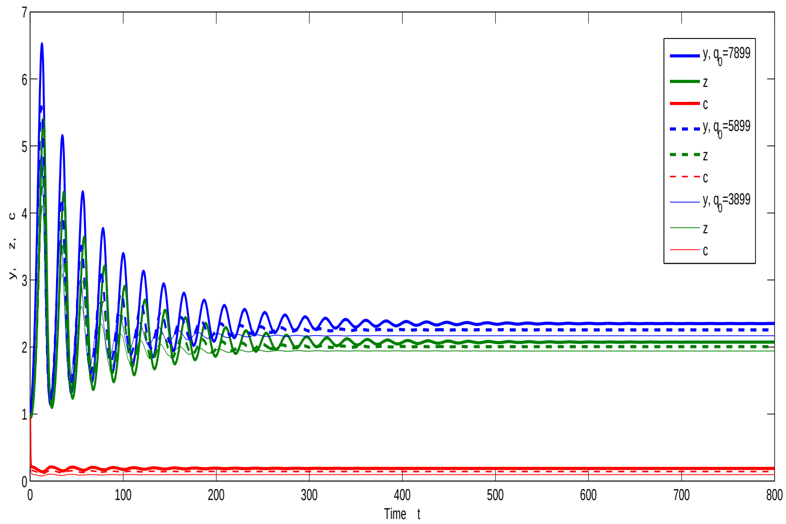

Figure 15. Contrast of y, z and c with respect to time t, for different values of q_0 in Submodel 2 of Model 2.

Table 2 and Fig.14, it might be seen from the calculations of the Submodel 2 of Model 2 that the system remains stable when the toxicant input rate q_0 increases, but the equilibrium point values decrease gradually.

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