Finding an Optimal Solution of Assignment Problem

Leena Jain[1*], Charanjit Singh[2]

[1]Department of Computer Applications, Global Group of Institutes, Amritsar, 143501(Punjab) India
[2]Department of Applied Science & Humanities, Global Group of Institutes, Amritsar, 143501(Punjab) India

[*]Corresponding Author: -
Prof. (Dr.) Leena Jain
Head-Department of Computer Applications
Global Group of Institutes, Amritsar (Punjab)
Email id: Leenajain79@gmail.com

Abstract:
Assignment model comes under the class of linear programming model which is the most used techniques of operations research, which looks alike with transportation model with an objective function of minimizing the time or cost of manufacturing the products by allocating one job to one machine or one machine to one job or one destination to one origin or one origin to one destination only. In this paper, we represent linear mathematical formulation of Assignment problem and solved using Lingo Software.

Keyword: Resource Allocation, Optimization Problem, Lingo Software, Assignment problem.

Introduction

The assignment problem is a special type of transportation problem, where the objective is to minimize the cost or time of completing a number of jobs by a number of persons. In other words, when the problem involves the allocation of *n* different facilities to *n* different tasks, it is often termed as an assignment problem. The assignment model is useful in solving problems such as, assignment of machines to jobs, assignment of salesmen to sales territories, travelling salesman problem, etc. It may be noted that with n facilities and n jobs, there are n! possible assignments. One way of finding an optimal assignment is to write all the n! possible arrangements, evaluate their total cost, and select the assignment with minimum cost. But, due to heavy computational burden this method is not suitable. In this paper, Assignment problem solved using Lingo Software. LINGO is a simple tool for utilizing the power of linear and nonlinear optimization to formulate large problems concisely, solve them, and analyze the solution. Optimization helps you find the answer that yields the best result; attains the highest profit, output, or happiness; or the one that achieves the lowest cost, waste, or discomfort. Often these problems involve making the most efficient use of your resources-including money, time, machinery, staff, inventory, and more. Optimization problems are often classified as linear or nonlinear, depending on whether the relationships in the problem are linear with respect to the variables.

Formulation of an assignment problem

Suppose a company has n persons of different capacities available for performing each different job in the concern, and there are the same numbers of jobs of different types. One person can be given one and only one job. The objective of this assignment problem is to assign n persons to n jobs, so as to minimize the total assignment cost. The cost matrix for this problem is given below figure1:

	Job				
Worker	j_1	j_2	----	j_n	a_i
i_1	x_{11} c_{11}	x_{12} c_{12}	----	x_{n1} c_{1n}	1
i_2	x_{21} c_{21}	x_{22} c_{22}	----	x_{2n} c_{2n}	1
----	----	----	----	----	--
i_n	x_{n1} c_{n1}	x_{n2} c_{n2}	----	x_{nn} c_{nn}	1
b_j	1	1	----	1	

Figure 1: Cost Matrix

To formulate the assignment problem in mathematical programming terms, we define the activity variables as

$$X_{ij} = \begin{cases} 1 \text{ if job j is performed by worker i} \\ 0 \text{ otherwise} \end{cases}$$

For i=1 to n & j=1 to n
In the above table, cij is the cost of performing j^{th} job by i^{th} person.

Optimization Model is

Minimize $C_{11}x_{11}+C_{12}x_{12}+\ldots\ldots.+C_{nn}x_{nn}$
$X_{i1}+X_{i2}+\ldots\ldots+X_{in}=1 \quad i=1,2,3,\ldots\ldots n$
$X_{1j}+X_{2j}+\ldots\ldots+X_{nj}=1 \quad j=1,2,3,\ldots\ldots n$
$X_{ij}=0$ or 1

Assumptions in Assignment Problem

- Number of jobs is equal to the number of machines or persons.
- Each man or machine is assigned only one job.
- Each man or machine is independently capable of handling any job to be done.
- Assigning criteria is clearly specified (minimizing cost or maximizing profit).

Literature Review

Branch Boundary Algorithm, Brute Force Algorithm, Hungarian Algorithm and heuristic algorithms (genetic algorithms, parallel auction algorithm, penalty method, and greedy method) are developed for solving Assignment. Among these, the Greedy algorithm is one of the most preferred methods because it is easy to apply and requires a small number of iterations. There are more than 10 algorithms that are similar in performances (and which one is the best depends on specific situations) in these days, and studies about this topic are still on-going [1]. Classical and heuristic algorithms for assignment problems are given in Table 1

Table 1: Algorithms for Assignment Problems

Classical Algorithms	Heuristic Algorithms
Simplex Method (LP)	OACE Algorithm
Transportation Method	Auction Algorithm
Brute Force Algorithm	Greedy Algorithm
The Branch Boundary Algorithm	Genetic Algorithm
Hungarian Algorithm	Harmony Search Algorithm
Lagrangian Relaxation Algorithm	Penalty Method

Among diverse approaches developed for assignment problem, Hungarian method [2,3], linear approaches [4], and the newer one- Auction algorithm [5] and heuristic algorithms are commonly used. The Branch Boundary Algorithm, reported by Breu and Burdet (1974), Wolsey and Nemhauser (1988), and Rijavec (1992) is the only known algorithm for solving the multidimensional assignment problem optimally [6, 7, 8]. For problems of any size, since the time required for computing the optimal solution grows exponentially with the size of the problem, this algorithm is impractical [9]. Since assignment problem in Operations Research is a main problem, a large number of studies have been conducted about this topic and various algorithms have emerged [10]. Aringhieri et al. (2015) presented a two-stage heuristic algorithm for the assignment problem. The algorithm was tested using real data collected from a state hospital in Genova, Italy. The results show that the proposed method performs well in terms of both solution quality and computation time [11]. When the computer solutions are taken into consideration with the classical solution methods, an increasing number of (n) and increasing processing time due to the increased memory requirement and high degeneration are encountered. Therefore, there are also special solution methods which are different from the classical 130 Different Approaches to Solution of The Assignment Problem Using R Program methods and use their characteristics. But none of these methods is as well-known as the Hungarian solution. This method is almost recognized enough to be referred to together with the assignment problem [12]. Aktel et al. (2017) have shown that metaheuristic algorithms for door assignment problems provide good results at a reasonable time for large-scale door assignment problems [13]. Seethalakshmy and Srinivasan (2018) discussed a methodology to determine the transportation problem in their study. In their study, they saw that transport could not be kept away from the assignment, in the end, proposed a new algorithm that combined the assignment in the transport [14]. Reyes et al. (2019) conducted a comprehensive literature review for assignment problems. In this study, they examined 71 representative articles and discussed solution methods. They have found that there are many solution methods for the assignment problem (classical, heuristic, meta-heuristic). They classified 71 representative papers according to solution methods, performance measures and constraints or concerns. The authors think that the current review is the most comprehensive and detailed study to date [15]. Various other linear network flow problems can be reformulated as assignment problems. To name a few the classical shortest path problems, there are max-flow problems, linear transportation problems, and linear minimum cost flow problems. Therefore, the solution of assignment problems is important. Comparison of computational complexity for different optimal assignment algorithms is given in Table 2 [16]. In this paper we construct linear mathematical formulation of assignment problem & then solved by Lingo Software.

Table 2: Comparison of the computational complexity for different optimal assignment algorithms

Algorithms for Assignment problem	Complexity order
Brute Force Method	$(n!)$
OACE Algorithm	$(n4)$
Hungarian Algorithm	$(n3)$
Auction Algorithm	$(n2)$
Greedy Algorithm	$(n2logn)$

Experimental Setup

A company has 4 machines to do 3 jobs. Each job can be assigned to one and only one machine. The cost of each job on a machine is given in the table 3.

Table 3: Cost Matrix

Jobs ↓ \ Machines →	W	X	Y	Z
A	**18**	**24**	**28**	**32**
B	**8**	**13**	**17**	**18**
C	**10**	**15**	**19**	**22**

What are the job assignments which will minimize the cost?
Source: Operations Research by Sharma, J. K. 5th edition Page 327, question 3

Mathematical Formulation

In this problem we have 4 machines and 3 jobs so this is the unbalanced assignment problem (No. of Machines≠ No. of Jobs). In this case we add dummy row D with cost of each 0 and balanced cost matrix is given in table 4.

Table 4: Balanced Cost Matrix

Jobs ↓ \ Machines →	W	X	Y	Z
A	**18**	**24**	**28**	**32**
B	**8**	**13**	**17**	**18**
C	**10**	**15**	**19**	**22**
D	**0**	**0**	**0**	**0**

Linear Mathematical formulation of above problem
Min Z=
$8*X_{11}+24*X_{12}+28*X_{13}+32*X_{14}+8*X_{21}+13*X_{22}+17*X_{23}+18*X_{24}+10*X_{31}+15*X_{32}+19*X_{33}++22*X_{34}+0*X_{41}+0*X_{42}+0*X_{43}+0*X_{44}$

Subject to Constraints:

$X_{11}+X_{12}+X_{13}+X_{14}=1$

$X_{21}+X_{22}+X_{23}+X_{24}=1$

$X_{31}+X_{32}+X_{33}+X_{34}=1$

$X_{41}+X_{42}+X_{43}+X_{44}=1$

$X_{11}+X_{21}+X_{31}+X_{41}=1$

$X_{12}+X_{22}+X_{32}+X_{42}=1$

$X_{13}+X_{23}+X_{33}+X_{43}=1$
$X_{14}+X_{24}+X_{34}+X_{44}=1$
where $X_{ij}=0$ or 1, i=1,2,3,4 & j=1,2,3,4

To solve the above linear formation of assignment problem, Software LINGO Version 11 was used. Figure 1 and Figure 2 illustrate the snap shots of construction of ILPP and generated solution by software LINGO/WIN32 19.0.55, LINDO API 13.0.4099.342, respectively.

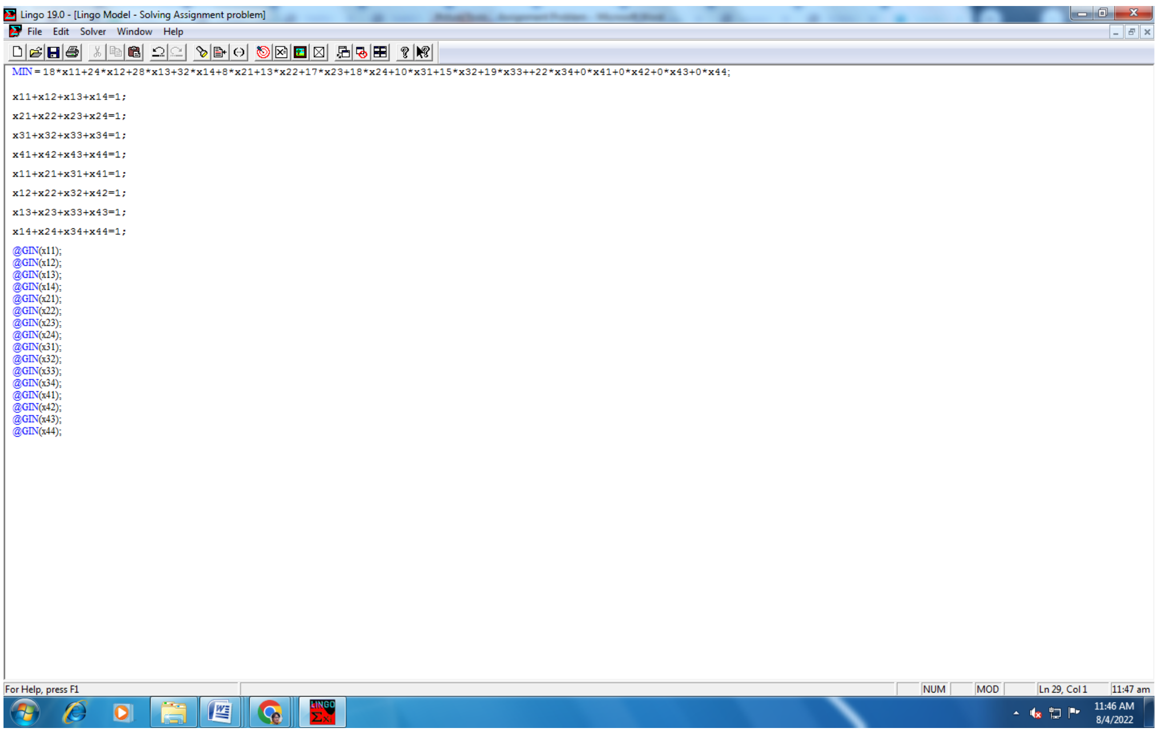

Figure 1: Formulation of Assignment Problem

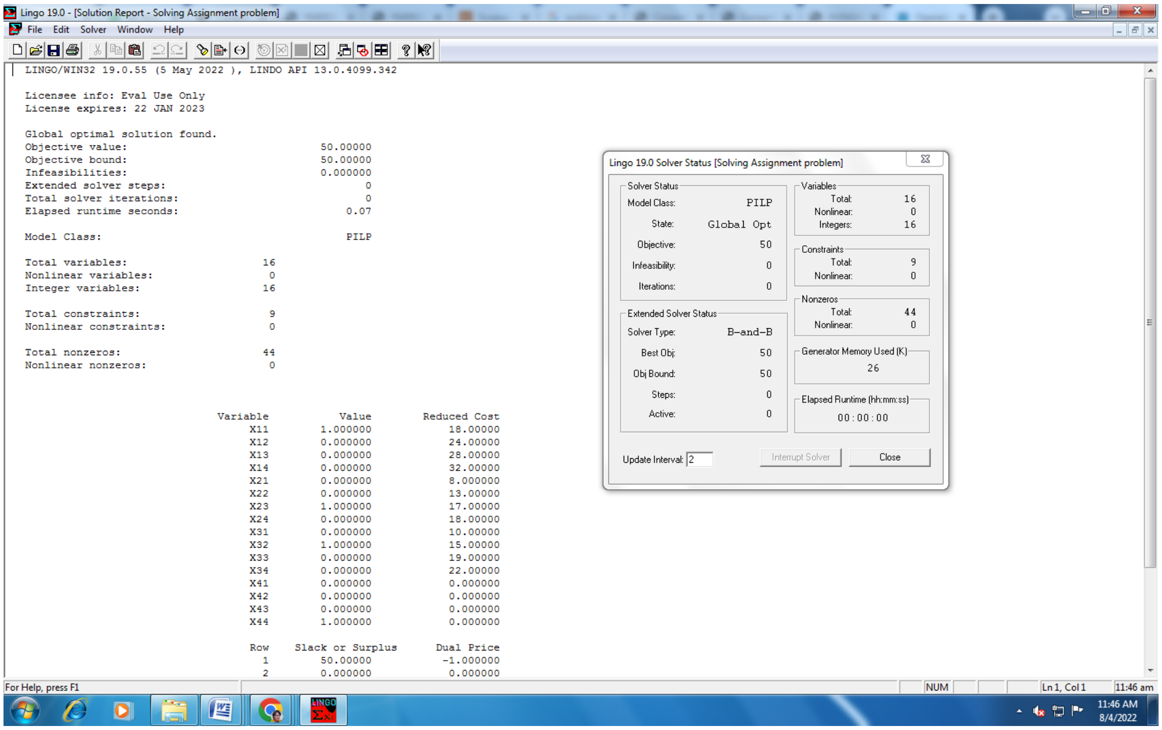

Figure 2: Solution of Assignment problem

Description of Lingo 19.0 Solver Status (Solving Assignment Problem)

Solver Status Box

The *Solver Status* box shows the current status of the solver. A description of the fields appears in the table 5 followed by a more in-depth explanation.

Table 5: Description of the field in Solver Status Box

Field	*Description*
Model Class	Displays the model's classification. Possible classes are "LP", "QP", "CONE", "NLP", "MILP", "MIQP", "MICONE", "MINLP","PILP", "PIQP", "PICONE", and "PINLP".
State	Gives the Status of the current solution. Possible states are "Global Optimum", "Local Optimum", "Feasible", "Infeasible", "Unbounded", "Interrupted", and "Undetermined".
Objective	Current value of the objective function.
Infeasibility	Amount constraints are violated by.
Iterations	Number of solver iterations.

Model Class Field

The *Model Class* field summarizes the properties of your model. The various classes you will encounter are listed in table 6 below:

Table 6: Various Model Class

Abbreviation	*Class*	*Description*
LP	Linear Program	All expressions are linear and the model contains no integer restrictions on the variables.
QP	Quadratic Program	All expressions are linear or quadratic and there are no integer restrictions.
CONE	Conic Program	The model is a conic (second-order cone) program and all variables are continuous.
NLP	Nonlinear Program	At least one of the relationships in the model is nonlinear with respect to the variables.
MILP	Mixed Integer Linear Program	All expressions are linear, and a subset of the variables is restricted to integer values.
MIQP	Mixed Integer Quadratic Program	All expressions are either linear or quadratic and a subset of the variables has integer restrictions.
MICONE	Mixed Integer Conic Program	The model is a conic (second-order cone) program, and a subset of the variables is restricted to integer values.
MINLP	Integer Nonlinear Program	At least one of the expressions in the model is nonlinear, and a subset of the variables has integer restrictions. ***In general, this class of model will be very difficult to solve for all but the smallest cases.***
PILP	Pure Integer Linear Program	All expressions are linear, and all variables are restricted to integer values.

PIQP	Pure Integer Quadratic Program	All expressions are linear or quadratic and all variables are restricted to integer values.
PICONE	Pure Integer Conic (Second-Order Cone) Program	The model is a conic (second-order cone) program, and all the variables are restricted to integer values.
PINLP	Pure Integer Nonlinear Program	At least one of the expressions in the model is nonlinear, and all variables have integer restrictions. ***In general, this class of model will be very difficult to solve for all but the smallest cases.***

In our case Software use **PILP Model** to solve the Assignment problem

State Field

When LINGO begins solving your model, the initial state of the current solution will be "Undetermined". This is because the solver has not yet had a chance to generate a solution to your model. Once the solver can no longer find better solutions to your model, it will terminate in either the "Global Optimum" or "Local Optimum" state. If your model does not have any nonlinear constraints, then any locally optimal solution will also be a global optimum. Thus, all optimized linear models will terminate in the global optimum state. If, on the other hand, your model has one or more nonlinear constraints, then any locally optimal solution may not be the best solution to your model. Finally, the "Interrupted" state will occur when you prematurely interrupt LINGO's solver before it has found the final solution to your model. In our case Software produces **Global Optimal solution** of the Assignment problem.

Objective Field

The *Objective* field gives the objective value for the current solution. If your model does not have an objective function, then "N/A" will appear in this field. In our case Software produces **Value of Objective function is 50** of the Assignment problem.

Infeasibility Field

The *Infeasibility* field lists the amount that all the constraints in the model are violated by. Keep in mind that this figure does not track the amount of any violations on variable bounds. Thus, it is possible for the *Infeasibility* field to be zero while the current solution is infeasible due to violated variable bounds. The LINGO solver may also internally scale a model such that the units of the *Infeasibility* field no longer correspond to the unscaled version of the model. In our case Software produces **infeasibility to 0** of the Assignment problem.

Iterations Field

The *Iterations* field displays a count of the number of iterations completed thus far by LINGO's solver. The fundamental operation performed by LINGO's solver is called an *iteration*. An iteration involves finding a variable, currently at a zero value, which would be attractive to introduce into the solution at a nonzero value. This variable is then introduced into the solution at successively larger values until either a constraint is about to be driven infeasible or another variable is driven to zero. At this point, the iteration process begins anew. In general, as a model becomes larger, it will require more iterations to solve and each iteration will require more time to complete.

Extended Solver Status Box

The *Extended Solver Status* box shows status information pertaining to several of the specialized solvers in LINGO. These solvers are:

- BNP Solver
- Branch & Bound Solver
- Global Solver and
- MUltistart Solver

The fields in this box will be updated only when one of these three specialized solvers is running. The fields appearing in the *Extended Solver Status* box are given in table 7:

Table 7: Field of Extended Solver Status box

Field	***Description***
Solver Type	The type of specialized solver in use, and will be either "B-and-B", "Global", "Multistart", or "BNP".
Best Obj	The objective value of the best solution found so far.
Obj Bound	The theoretical bound on the objective.
Steps	The number of steps taken by the extended solver.
Active	The number of active sub problems remaining to be analyzed.

Solver Type Field

This field displays either "BNP", "B-and-B", "Global", or "Multistart", depending on the specialized solver in use. LINGO employs a strategy called ***branch-and-bound*** to solve models with integer restrictions. Branch-and-bound is a systematic method for implicitly enumerating all possible combinations of the integer variables.

Best Obj and Obj Bound Fields

The *Best Obj* field displays the best feasible objective value found so far. *Obj Bound* displays the bound on the objective.

Steps Field

The information displayed in the *Steps* field depends on the particular solver that is running. The table 8 represent different steps field for different solver.

Table 8: Different steps field for different solver

Solver	***Steps Field Interpretation***
BNP	Number of branches in the branch-and-bound tree.
Branch-and-Bound	Number of branches in the branch-and-bound tree.
Global	Number of subproblem boxes generated.
Multistart	Number of solver restarts.

Active Field

This field pertains to the BNP, branch–and–bound and global solvers. It lists the number of open sub problems remaining to be evaluated. The solver must run until this valve goes to zero.

Variables Box

The *Variables* box shows the total number of variables in the model. The *Variables* box also displays the number of the total variables that are *nonlinear*. The *Variables* box in the *solver status* window also gives you a count of the total number of *integer* variables in the model. In general, the more nonlinear and integer variables your model has, the more difficult it will be to solve to optimality in a reasonable amount of time. Pure linear models without integer variables will tend to solve the fastest.

Constraints Box

The *Constraints* box shows the total constraints in the expanded model and the number of these constraints that are *nonlinear*. A constraint is considered nonlinear if one or more variables appear nonlinearly in the constraint.

Nonzeroes Box

The *Nonzeros* box shows the total *nonzero coefficients* in the model and the number of these that appear on nonlinear variables. In a given constraint, only a small subset of the total variables typically appears. The implied coefficient on all the non-appearing variables is zero, while the coefficients on the variables that do appear will be nonzero. Thus, we can view the total nonzero coefficient count as a tally of the total number of times variables appear in all the constraints. The nonlinear nonzero coefficient count can be viewed as the number of times variables appear nonlinearly in all the constraints.

Generator Memory Used Box

The *Generator Memory Used* box lists the amount of memory LINGO's model generator is currently using from its memory allotment.

Elapsed Runtime Box

The *Elapsed Runtime* box shows the total time used so far to generate and solve the model. This is an elapsed time figure and may be affected by the number of other applications running on your system.

Findings of Our Problem

We get optimal solution of above Assignement Problem in .06 Seconds (Elapsed Runtime) with Optimal Value 50 which is equivalent to optimal solutions of Hungarian assignment method. Lingo software use Model class Pure Integer Linear Program(PILP) and solved problem using Branch & Bound Method.

$X_{11}=X_{23}=X_{32}=X_{44}=1$

$X_{12}=X_{13}=X_{14}=X_{21}=X_{22}=X_{24}=X_{31}=X_{33}=X_{34}=X_{42}=X_{43}=X_{41}=0$

Optimal Assignements of above problem are

A->W With Cost 18

B->Y With Cost 17

C->X With Cost 15

D->Z With Cost 0 (Dummy Assignment)

Total Minimum Cost=18+17+15+0=50

Conclusion

In this paper, we presented linear formulation of Assignment problem & then solve by Lingo Software. And we get the optimal solution in less than one second which is same as the optimal solutions of Hungarian assignment method. We also discussed detail description of Lingo 19.0 Solver Status.

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