The Solution of Integrable nonlinear evolution equation with time-dependent coefficient

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Abstract

In this paper, we consider the evolution equation $u_t = [m + k\psi(u)]u, \quad t > 0$ where t represents dimensionless time and $\psi(u)$ is given function of u. We focus attention on the case when $\psi(u) = k(u - a)(u - 1)$, while m and k are constants.

Keywords

Evolution equation, Numerical solution, Initial Value Problem

1. Introduction

Differential equations are one of the major branches in the mathematics which solves many real-life time problems (Hermann, Martin and Saravi, Masoud, 2014). It also plays vital role in the recent developments of some main branches like earth science, economic, medicines, etc., There are two major parts of differential equations are developed as linear and non-linear equations.

When we consider non-linear differential equations, which are bit difficult to solve analytically. In such cases, we opt some numerical techniques to get a true solution by an approximation.

The evolution of any system depends on time variable which is in the form of $\frac{du}{dt} = g(u)$. The solution u belongs to a function space with the corresponding time. A differential equation which involves time derivatives is referred as an evolution equation.

The classical non-steady heat equation and wave equation are the best and simple examples of evolution equations. We can find many different forms of equations in different fields as Schrodinger equation in Quantum mechanics, Chorin and Marsden has found an equation on flow of blood in a human blood. (Segal, 1969) has introduced an equation on thermal convection which effects of side walls. Davey Stewartson (Hocking, L. M. and Stewartson, K. , 1971)introduced the concept of non-linear evolution of water waves in one dimension etc.,

Let $x(t)$ be a real valued function on t and $\frac{dx}{dt}$ to be a velocity representing the flow on axis of x. Consider $f(x)$ which is a smooth and real valued function. Let us imagine a fluid flowing along with the x -axis. If x -axis represents one dimensional space, then this imaginary fluid is called the phase fluid (Hermann, Martin and Saravi, Masoud, 2014).The sign of $\frac{dx}{dt}$ determines the sign of the function $f(x)$. The fluid flow is to the right when $f(x) > 0$ and to left where $f(x) < 0$ and the phase point which is found by considering an imaginary fluid particle with initial position as x_0 at $t = 0$.

Let us observe how this particle is carried along in its flow. When t increases, the phase point moves along the x -axis according to $x(t)$ which is known as trajectory (Hermann, Martin and Saravi, Masoud, 2014) of the fluid particle. The equilibrium points which found from $f(x_E) = 0$ by the fixed point x_E.

The stable equilibrium is defined based on considering small perturbations. When consider bifurcation when the dependence of x on a parameter is varied dynamically with qualitative changes in the system. This parameter values at which bifurcation happening are known as bifurcation points (Wilford Zdunkowski and Andreas Bott , 2003) .

2. Equilibrium solutions and their Stability

We first consider some general properties of an example which is of relevance to nerve impulse transmission (Johns, D.S and Sleeman,D , 2003)

$$u_t = mu + k(u-a)(u-1)u, \quad t > 0. \tag{1}$$

subject to condition

$$u(0) = u_0 \tag{2}$$

Here a is positive constant, which without any loss of generality; we can assume that it is less than unity, since for $a > 1$ the function $u(t)$ can be scaled by division by a with a redefinition of parameters m and k with the new parameter a being the reciprocal of the original a.

We will examine the stability of the solution (1). If m and k are real, then let

$$\eta = \frac{m}{k} \tag{3}$$

and the bifurcation curve in the (η, u) plane is given by

$$\eta u + (u-a)(u-1)u = 0 \tag{4}$$

This gives the equilibrium solution

$$u_{c1} = 0 \tag{5}$$

and two more equilibrium solutions

$$u_{c\pm} = \frac{1+a}{2} \pm \sqrt{\frac{(1-a)^2}{4} - \eta} \tag{6}$$

Provided

$$\eta < \frac{(1-a)^2}{4} \tag{7}$$

The bifurcation curve is shown in Figure 1 when (1.6) is satisfied. If

$$\lambda^2 = \eta - \frac{(1-a)^2}{4} \tag{8}$$

Then we need to consider three cases:

1. $\lambda^2 = 0$
2. $\lambda^2 < 0$
3. $\lambda^2 > 0$

2.1. Case 1

It is associated with equilibrium solutions with one at $u = u_{c1}$ and $u_{c2} = \frac{1+a}{2}$. At u_{c1}, the derivative $\frac{du}{dt}$ changes from negative to positive as u increases from negative values to positive values, assuming $k > 0$. The reverse happens if $k < 0$. Thus $u = u_{c1}$ is an unstable solution. At $u = u_{c2}$, $\frac{du}{dt} > 0$ on either side of $u = u_{c2}$. This is because $u_{c2}$is an equilibrium solution and a bifurcation point and its stability is affected by bifurcation property. The equilibrium solution u_{c2} is therefore stable is given in Figure 1(b).

2.2. Case 2

In this case, the condition (7) is satisfied and there are three equilibrium solutions. It is clear Figure 2(a) that the u_{c1} and u_{c+} are unstable while u_{c-} is stable.

2.3. Case 3

In this case, the condition (7) is not obeyed and hence there is only equilibrium solution u_{c-} which is unstable is given for illustration in Figure 2(b).

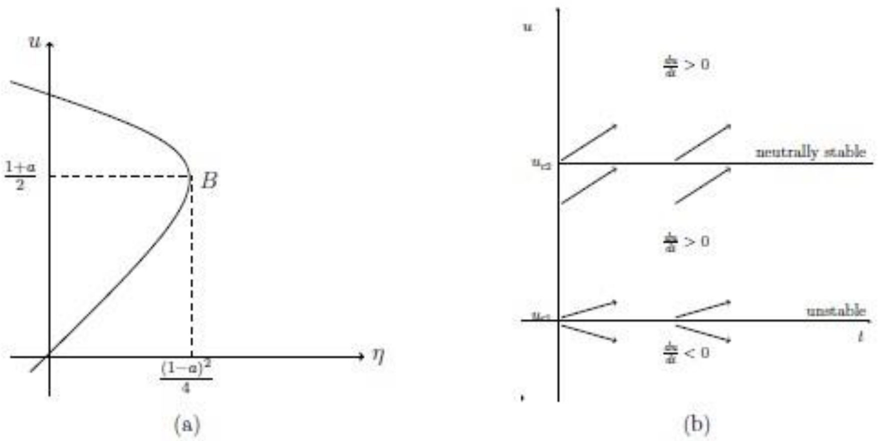

Figure 1 (a) A schematic diagram for the bifurcation curve of the system (1) and (2). (b) Case 1 when $\lambda^2 = 0$ where B is a bifurcation point.

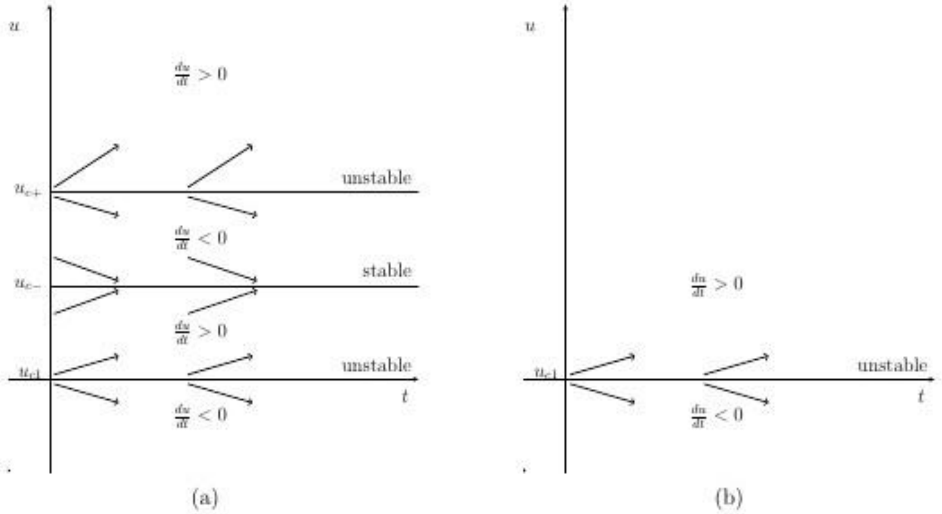

Figure 2 (a) Case 2 when $\boldsymbol{\lambda^2 < 0}$. ***. (b) Case 3 when*** $\boldsymbol{\lambda^2 > 0}$.

3. Integration the equation (1)

We now proceed to integrate the equation (1). Using partial fractions, we find that

$$(ak+m)t = \frac{1}{2}\ln\left\{\frac{u^2\left[u_0^2-(1+a)u_0+a+\frac{m}{k}\right]}{u_0^2\left[u^2-(1+a)u_0+a+\frac{m}{k}\right]}\right\} + \frac{1+a}{2}\int_{u_0}^{u}\frac{dv}{\left(v-\frac{1+a}{2}\right)^2+\frac{m}{k}-\frac{(1-a)^2}{4}} \quad (9)$$

The evaluation of last integral depends on the sign of the term $\lambda^2 = \frac{m}{k} - \frac{(1-a)^2}{4}$ in the denominator. Therefore, this delineates three cases: $\lambda^2 = 0$, $\lambda^2 > 0$ and $\lambda^2 < 0$, which we need to investigate each separately.

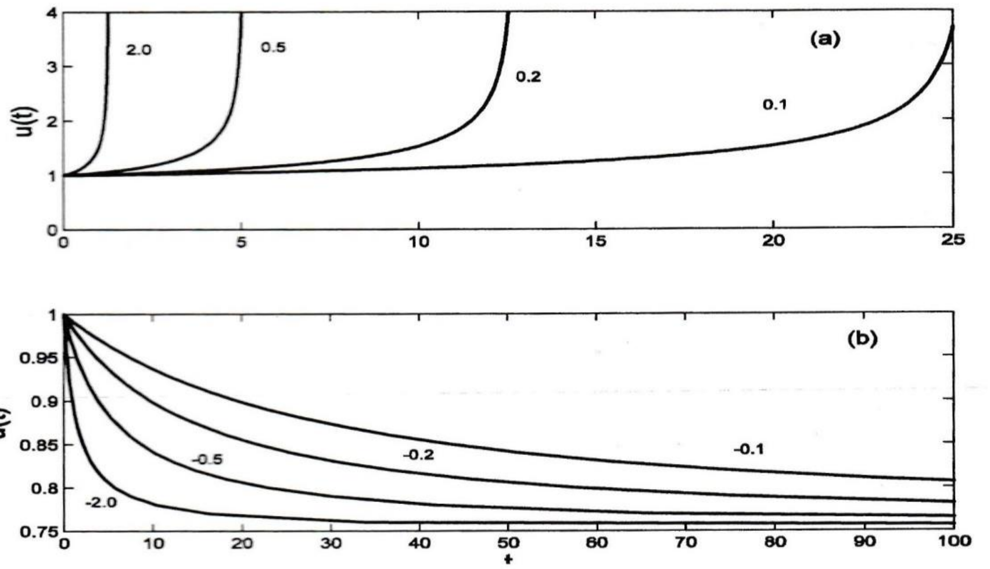

Figure 3 Graphs of the numerical solutions of (1) in the (t, u) plane when a=0.5 . (a) Represent the solutions when k=0.1, 0.2, 0.5 and 2. (b) Represent the solutions when k=-0.1, -0.2, -0.5 and -2.

3.1 When $\boldsymbol{\lambda^2 = 0}$

In this case, we can integrate the second expression on the right-hand side of (9) to obtain

$$(ak+m)t = \frac{1}{2}\ln\left\{\frac{u^2\left[u_0^2-(1+a)u_0+a+\frac{m}{k}\right]}{u_0^2\left[u^2-(1+a)u_0+a+\frac{m}{k}\right]}\right\} + \frac{1+a}{2}\left[\frac{\mathbf{1}}{\boldsymbol{2u-(1+a)}} - \frac{\mathbf{1}}{\boldsymbol{2u_0-(1+a)}}\right] \quad (10)$$

This expression gives the inverse $t(u) = t$ for the behaviour of the function $u(t)$ on t but it is very difficult to see the dependence of u on t. The expression has been evaluated numerically for various of real parameters m, a and k subject to the condition (2). A sample of results is shown in Figure 3.

3.2 When $\boldsymbol{\lambda^2 > 0}$

In this case, the integration of (9) leads to

$$(ak+m)t = \frac{1}{2}\ln\left\{\frac{u^2\left[u_0^2-(1+a)u_0+a+\frac{m}{k}\right]}{u_0^2\left[u^2-(1+a)u_0+a+\frac{m}{k}\right]}\right\} + \frac{1+a}{2\lambda}\boldsymbol{tan^{-1}}\left[\frac{\mathbf{4}\lambda(u-u_0)}{\mathbf{4}\lambda^2+(\boldsymbol{2u-1-a})(\boldsymbol{2u_0-1-a})}\right] \quad (11)$$

The solution here is illustrated in Figure 4 and 5. We note the solution decays with time if $k < 0$ and the reverse happens if $k > 0$.

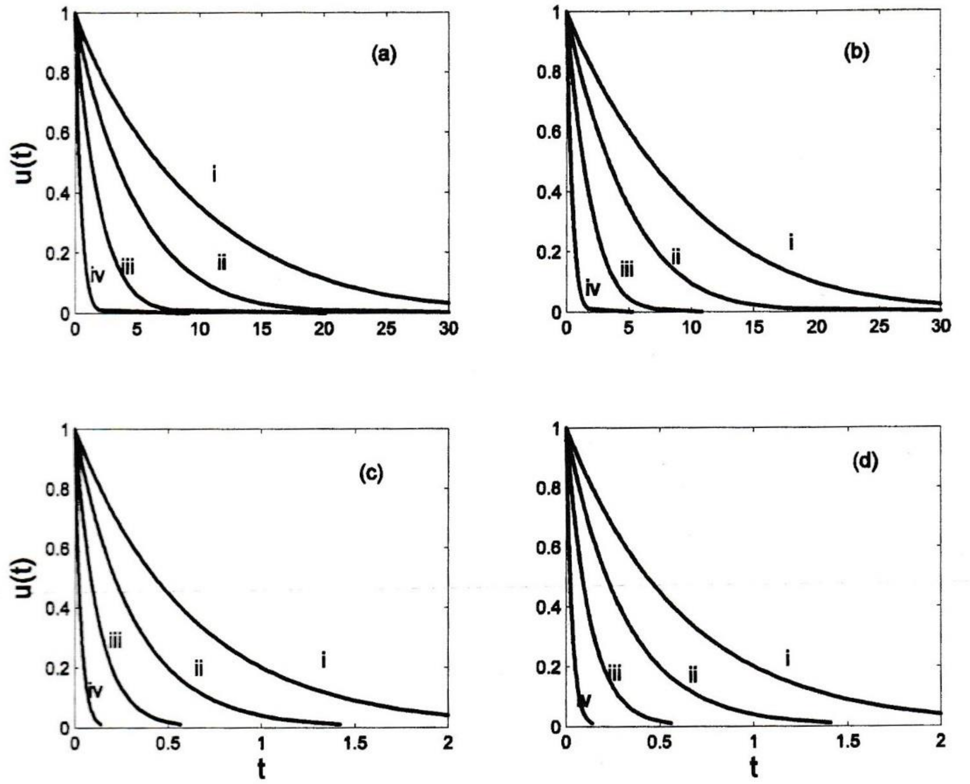

***Figure 4** Graphs of the numerical solutions of (1) in the (t, u) plane for the case $\lambda^2 > 0$. The curves (i)-(iv) correspond to k=-0.1, -0.2, -0.5 and -2, respectively. The subfigures (a), (b), (c) and (d) correspond to (a, λ) taking the pair of values (0.2,1.0), (0.5,1.0), (0.2,4.0) and (0.5,4.0), respectively. The decay of the solution is very strongly increased by the increase in λ.*

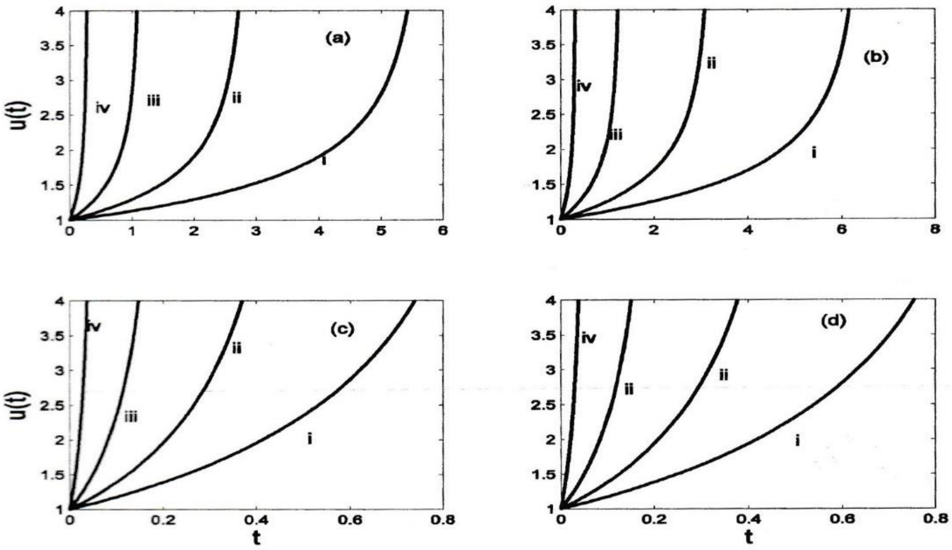

***Figure 5** Graphs of the numerical solutions of (1) in the (t, u) plane for the case $\lambda^2 < 0$. The curves (i)-(iv) correspond to k=0.1, 0.2, 0.5 and 2 , respectively. The subfigures (a), (b), (c) and (d) correspond to correspond to (a, λ) taking the pair of values (0.2,1.0), (0.5,1.0), (0.2,4.0) and (0.5,4.0), respectively. The growth of the solution is very strongly increased by the increase in λ.*

3.3 When $\lambda^2 < 0$

In this case, the solution of (\ref{eqn41}) in the form

$$(ak+m)t = \frac{1}{2}\ln\left\{\frac{u^2\left[u_0^2-(1+a)u_0+a+\frac{m}{k}\right]}{u_0^2\left[u^2-(1+a)u_0+a+\frac{m}{k}\right]}\right\} + \frac{1+a}{4\Gamma}\boldsymbol{ln}\left[\frac{(2u-1-a-2\Gamma)(2u_0-1-a+2\Gamma)}{(2u-1-a+2\Gamma)(2u_0-1-a-2\Gamma)}\right] \quad (12)$$

in which

$$\Gamma = \left(-\frac{m}{k}+\frac{(1-a)^2}{4}\right)^{\frac{1}{2}} \quad (13)$$

We plot the numerical solution of (1) with above case in Figures 6 and 7.

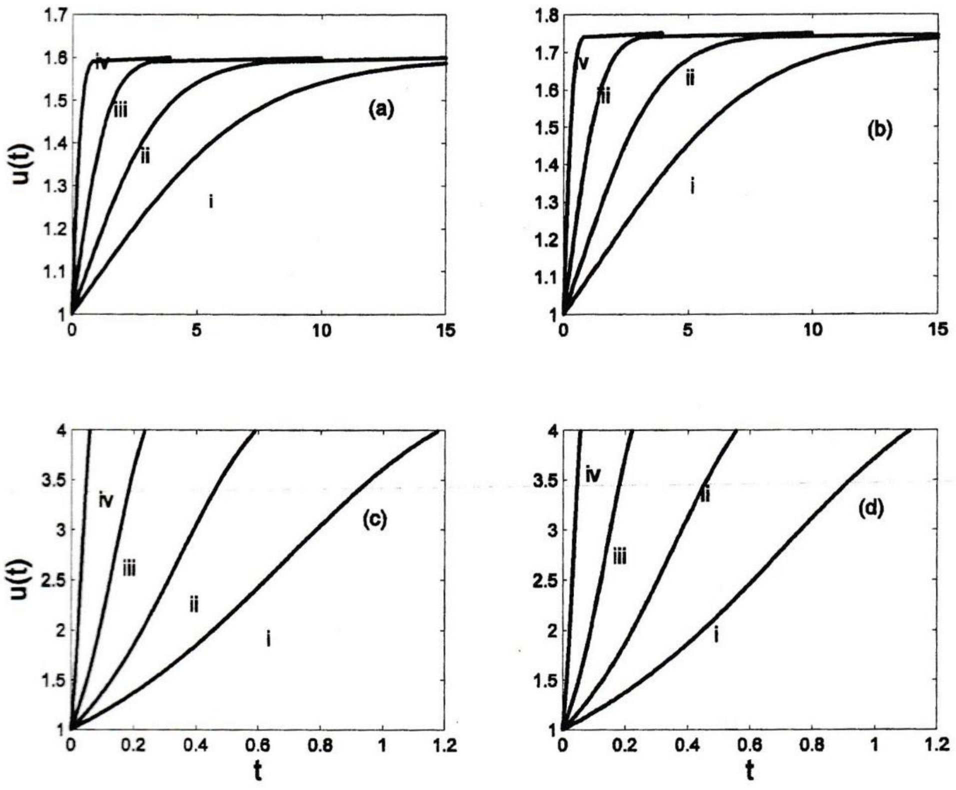

Figure 6 Graphs of the numerical solutions of (1) in the (t,u) plane when $\Gamma^2 = -\frac{m}{k}+\frac{(1-a)^2}{4}$. The curves labeled (i)-(iv) refer to k=-0.1, -0.2, -0.5 and -2 , respectively. The subfigures (a), (b), (c) and (d) correspond to (a, Γ)= (0.2,1.0), (0.5,1.0), (0.2,4.0) and (0.5,4.0), respectively.

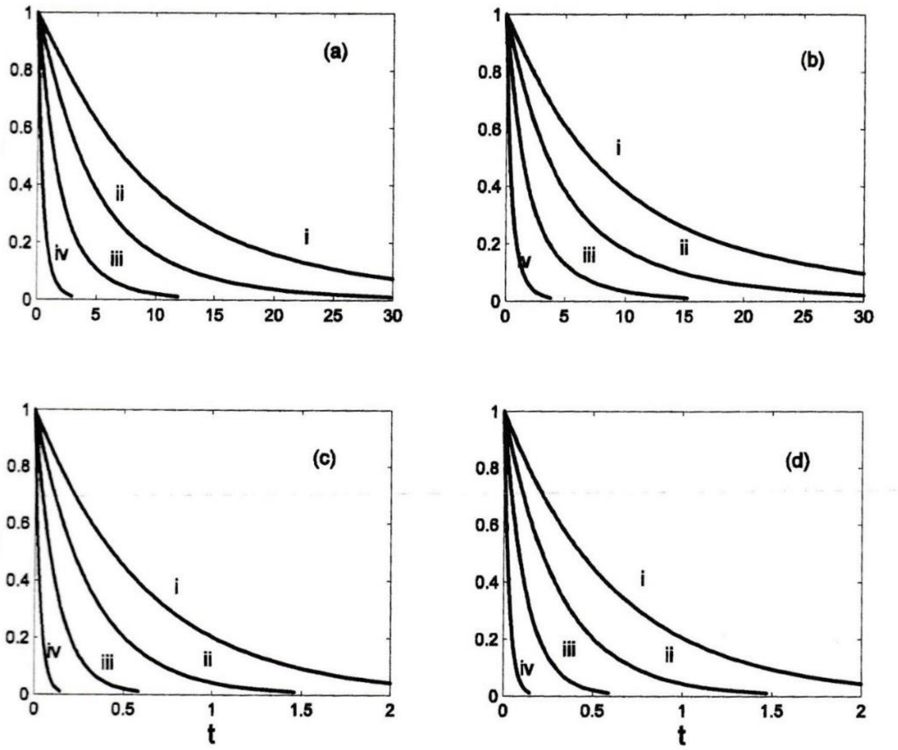

Figure 7 Graphs of the numerical solutions of (1) in the (t,u) plane when $\boldsymbol{\Gamma^2 = -\frac{m}{k} + \frac{(1-a)^2}{4}}$*. The curves labeled (i)-(iv) refer to k=0.1, 0.2, 0.5 and 2 , respectively. The subfigures (a), (b), (c) and (d) correspond to (a, Γ)= (0.2,1.0), (0.5,1.0), (0.2,4.0) and (0.5,4.0), respectively.*

5. Conclusions

In this paper, we discuss the initial value problem

$$u_t = mu + k(u-a)(u-1)u, \quad t > 0.$$

in which m, k, a and u_0 are constants with $a < 1$. This equation is of particular relevance to nerve impulse transmission. We have now completed the detailed of the nature of the solution to this equation depends on the quantity

$$\lambda^2 = \frac{m}{k} - \frac{(1-a)^2}{4}$$

three cases were isolated corresponding to the situations (i) $\lambda^2 = 0$, (ii) $\lambda^2 < 0$, and (iii) $\lambda^2 > 0$.. The solution was obtained in each case and illustrated graphically.

In addition to finding the solution of this equation, we have also discussed the stability of that solution. In order to find the relationship between the different solutions, we have examined the stability in relation to the bifurcation properties of the equation. It is shown that the equation (1) has either one equilibrium solution (bifurcation point), two or three depending on whether $\sqrt{\frac{(1-a)^2}{4} - \frac{m}{k}}$ is positive, zero or negative, when m and k are real. Every case was investigated, and the stability found.

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